Krasnoyarsk state university

Chair of applied mathematics

Risk theory

Topic: Correspondences

Lecture for students of math department of KSU

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Last modified 2/5/2002

Krasnoyarsk 2002

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Abstract

The concept of correspondence that is a generalization of the function concept is being considered in this lecture, as well as related polar and component concepts. Properties of these objects are studied and their connection to relation concept is considered. The lecture is essentially based on results presented in [1].

1 Correspondence

Definition 1.1 *Let X, Y be arbitrary sets. Correspondence $\Phi : X \to Y$ is any subset $\Phi \subseteq X \times Y$ of cartesian product X by Y.*

In particular, if any element $x \in X$ appears at most in one pair $(x, y) \in \Phi$, we get usual single-valued mapping from X to Y, or a *function.* Thus correspondence is a generalization of a function concept.

Consider an example of correspondence. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3, y_4\}$. Cartesian product $X \times Y$ is shown below as a rectangular matrix with elements of $\Phi = \{(x_1, y_1), (x_2, y_1), (x_2, y_2), (x_2, y_3)\}$ shown as bullets.

y_4	○	○	○
y_3	○	●	○
y_2	○	●	○
y_1	●	●	○
	x_1	x_2	x_3

Figure 1: Example of correspondence

An *image* $\Phi(A)$ of a subset $A \subseteq X$ under correspondence Φ is a set

$$\Phi(A) = \{y \in Y |\ \exists x \in A : \ (x, y) \in \Phi\}$$

of points $y \in Y$, that appear in Φ together with some $x \in A$. For $x \in X$ we will denote an image of a singleton $\{x\}$ also by $\Phi(\{x\}) = \Phi(x)$.

x_3	○	○	○	○
x_2	●	●	●	○
x_1	●	○	○	○
	y_1	y_2	y_3	y_4

Figure 2: Example of inverse correspondence

An *inverse* correspondence $\Phi^{-1} : \ Y \to X$ is a subset $\Phi^{-1} \subseteq Y \times X$ defined as follows:

$$\Phi^{-1} = \{(y, x) \in Y \times X |\ (x, y) \in \Phi\}.$$

The inverse correspondence for that shown in figure 1 is presented in figure 2, the latter obviously being a transpose of the figure 1. Image $\Phi^{-1}(B)$ of a set $B \subseteq Y$ under correspondence Φ^{-1} is also called a *proimage* of B under correspondence Φ. We will use the same agreement for proimages of singletons $\{y\}$ as above: $\Phi^{-1}(\{y\}) = \Phi^{-1}(y),\ y \in Y$.

The concept of correspondence is convenient in the sense that inverse correspondence always exists, and $(\Phi^{-1})^{-1} = \Phi$.

As can be seen from examples, some elements $x \in X$ may possess empty images under Φ, and similarly, for some $y \in Y$ it is possible that $\Phi^{-1}(y) = \emptyset$. This leads to concepts of *domain* $\mathcal{D}(\Phi)$ and *region* $\mathcal{R}(\Phi)$ of a correspondence Φ:

$$\mathcal{D}(\Phi) = \{x \in X |\ \exists y \in Y : \ (x, y) \in \Phi\}, \quad \mathcal{R}(\Phi) = \mathcal{D}(\Phi^{-1}).$$

In examples of figures 1, 2 one has $\mathcal{D}(\Phi) = \mathcal{R}(\Phi^{-1}) = \{x_1, x_2\}$, $\mathcal{R}(\Phi) = \mathcal{D}(\Phi^{-1}) = \{y_1, y_2, y_3\}$.

Sets X, Y are also called *departure* and *arrival* domains of Φ, respectively. In most cases without loss of generality one can get rid of "extra" points and let $X = \mathcal{D}(\Phi)$, $Y = \mathcal{R}(\Phi)$; in what follows it is always assumed.

Let $Z \subseteq X$. A *restriction* of Φ to Z is a correspondence $\Phi|_Z : Z \to \Phi(Z)$, defined by $\Phi|_Z = \{(x, y) \in \Phi |\ x \in Z\}$. In example shown in figure 1, we have $\Phi|_{\{x_1\}} = \{(x_1, y_1)\}$.

Consider two correspondences $\Phi : X \to Y$ and $\Psi : Y \to Z$. Their *composition* $\Theta = \Psi \circ \Phi$ is a correspondence $\Theta : X \to Z$, defined by

$$\Theta = \{(x, z) \in X \times Z |\ \exists y \in Y : (x, y) \in \Phi,\ (y, z) \in \Psi\}.$$

Some properties of correspondence are presented below; their proof is left to reader as exercise. We use the following notation: $\Phi : X \to Y$, $\Psi : Y \to Z$, $\Theta : Z \to U$ are correspondences between specified sets, $\{A_\lambda,\ \lambda \in \Lambda\}$ is any family of subsets of X, indexed by elements of a set Λ. For any set Z we denote $I_Z = \{(z, z),\ z \in Z\}$ the "main diagonal" of cartesian product $Z \times Z$.

1. Monotonicity by inclusion

$$A \subseteq B \subseteq X \implies \Phi(A) \subseteq \Phi(B).$$

2. Image of union and intersection

$$\Phi\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) = \bigcup_{\lambda\in\Lambda} \Phi\left(A_\lambda\right), \quad \Phi\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) \subseteq \bigcap_{\lambda\in\Lambda} \Phi\left(A_\lambda\right).$$

3. Image of a set via images of its elements

$$\Phi(A) = \bigcup_{x\in A} \Phi(x), \quad A \subseteq X.$$

4. Image of a composition

$$(\Psi \circ \Phi)(A) = \Psi(\Phi(A)).$$

5. Associativity of composition

$$\Theta \circ (\Psi \circ \Phi) = (\Theta \circ \Psi) \circ \Phi.$$

6. Domains and regions of mutually inverse correspondences.

$$\mathcal{D}(\Phi) = \mathcal{R}(\Phi^{-1}), \ \ \mathcal{R}(\Phi) = \mathcal{D}(\Phi^{-1}).$$

7. Composition of direct and inverse correspondence.

$$\Phi^{-1} \circ \Phi \supseteq I_X.$$

8. Representation of composition

$$\Theta \circ \Psi \circ \Phi = \bigcup_{(y,z)\in\Psi} \left(\Phi^{-1}(y) \times \Theta(z)\right).$$

2 Polar

Let $\Phi: \ X \to Y$ be a correspondence.

Definition 2.1 *Polar of* Φ *is a mapping* $\pi_\Phi: \ 2^X \ \to \ 2^Y$*, that transforms a set* $A \subseteq X$ *to a set*

$$\pi_\Phi(A) = \{y \in Y | \ \Phi^{-1}(y) \supseteq A\}.$$

If correspondence Φ is specified, it is convenient to write just π instead of π_Φ; besides that, we will use a short form for singletons $\{x\}$: $\pi(\{x\}) = \pi(x), \ x \in X$.

The polar of inverse correspondence Φ^{-1} is called inverse polar of Φ and is denoted by $\pi^{-1} = \pi_\Phi^{-1} = \pi_{\Phi^{-1}}$.

Below is a list of properties of polar, that reader may prove as exercise.

1. Polar of a singleton

$$\pi(x) = \Phi(x).$$

2. Polar of a set via polars of its elements

$$\pi(A) = \bigcap_{x \in A} \pi(x), \quad A \subseteq X.$$

3. Polar of a union

$$\pi\left(\bigcup_{\lambda \in \Lambda} A_\lambda\right) = \bigcap_{\lambda \in \Lambda} \pi\left(A_\lambda\right)$$

4. Monotonicity of polar by inclusion

$$A \subseteq B \subseteq X \implies \pi(A) \supseteq \pi(B).$$

5. Relation between direct and inverse polars

$$A \subseteq X,\ B \subseteq Y,\ A \times B \subseteq \Phi \implies A \subseteq \pi^{-1}(B), \quad B \subseteq \pi(A).$$

6. Composition of direct and inverse polars

$$A \subseteq X,\ B \subseteq Y \implies A \subseteq \pi^{-1}(\pi(A)), \quad B \subseteq \pi(\pi^{-1}(B)).$$

3 Separation theorem

Not each subset $B \subseteq Y$ may be a value of polar π, similarly, not each subset $A \subseteq X$ may be a value of π^{-1}. Next theorem provides a characterization of sets that are values of π^{-1}. Due to symmetry this theorem is easily stated for direct polar π as well.

Theorem 3.1 *Let $A \subseteq X$ be a fixed subset of X. A subset $B \subseteq Y$ such that $A = \pi^{-1}(B)$ exists if and only if for each $x \in X \setminus A = A^c$ there exists an element $y = y_x \in Y$ such that $\pi^{-1}(y_x) \supseteq A$ and $x \notin \pi^{-1}(y_x)$ (this also clearly implies $A = \pi^{-1}(\pi(A))$).*

Remark 3.1 *This theorem recalls separation theorems of convex analysis: value of polar π^{-1} at a point y_x "separates" a point x and a set A.*

Proof. **Sufficiency**. Define a set B as follows:

$$B = \bigcup_{x \in A^c} \{y_x\},$$

and show that $\pi^{-1}(B) = A$. Indeed, using the 3rd property of polar, we get

$$\pi(B) = \pi\left(\bigcup_{x \in A^c} \{y_x\}\right) = \bigcap_{x \in A^c} \pi(y_x) \supseteq A,$$

since by condition each set $\pi(y_x),\ x \in A^c$ contains A. If the last intersection was wider than A, it should contain an $x \notin A$ such that there exists y_x such that $x \notin \pi^{-1}(y_x)$: a contradiction to B construction. Thus, $\pi^{-1}(B)$ coincides with A, and sufficiency is established.

Necessity. Let there exists a $B \subseteq Y$ such that, $A = \pi^{-1}(B)$, or in other words

$$A = \{x \in X |\ \Phi(x) \supseteq B\}.$$

This implies that for any $x \notin A$ we have $\Phi(x) \not\supseteq B$, that is, there exists $y = y_x \in B$ such that $y_x \notin \Phi(x)$, or, equivalently $x \notin \Phi^{-1}(y_x) = \pi^{-1}(y_x)$. Besides that, due to monotonicity $\pi^{-1}(y_x) \supseteq \pi^{-1}(B) = A$. The proof is complete. $\diamond$

4 Component

Let $\Phi: X \to Y$ be a correspondence from X to Y, and $\pi: 2^X \to 2^Y$ be its polar. As was mentioned before, for any $A \subseteq X$ the following is true:

$$A \subseteq \pi^{-1}(\pi(A)). \tag{1}$$

Denote $\rho = \pi^{-1} \circ \pi$ a composition. Due to monotonicity of polars π, π^{-1} we have $\rho : 2^X \to 2^X$ and

$$A \subseteq B \subseteq X \implies \rho(A) \subseteq \rho(B). \tag{2}$$

Sets for which (1) turns to equality play a special role, they are called *components* of a correspondence and in a sense they are "fixed points" of ρ.

Definition 4.1 *A set $H \subseteq X$ is called a component of a correspondence Φ, if $H = \pi^{-1}(\pi(H))$.*

By the theorem 3.1 a set $H \subseteq X$ is a component if and only if $H = \pi^{-1}(B)$ for some $B \subseteq Y$. This implies that $\pi^{-1}(\pi(A))$ is the least (by inclusion) component containing $A \subseteq X$. Denote it by $\overline{A}$:

$$\overline{A} = \pi^{-1}(\pi(A)).$$

Let us prove some properties of components.

Proposition 4.1 *Intersection of any family of components is a component itself.*

Proof. Indeed, let $\{H_\lambda,\ \lambda \in \Lambda\}$ be a family of components of a correspondence Φ. By property 3 of section 2 we have

$$H = \bigcap_{\lambda\in\Lambda} H_\lambda = \bigcap_{\lambda\in\Lambda} \pi^{-1}(\pi(H_\lambda)) = \pi^{-1}\left(\bigcup_{\lambda\in\Lambda} \pi(H_\lambda)\right),$$

thus theorem 3.1 implies that H is a component. $\diamond$

Proposition 4.1 allows defining a least component that is an intersection of all components of a correspondence Φ. It clearly coincides with $\pi^{-1}(Y)$ and is called a *center* of correspondence.

Proposition 4.2 *X is a (largest) component of Φ.*

Proof follows from $X = \pi^{-1}(\emptyset)$ and theorem 3.1. $\diamond$

If H is a component of Φ, then $H' = \pi(H)$ is clearly a component of Φ^{-1}; the latter is called a *complement* of H. Denote $\mathcal{K}_\Phi \subseteq 2^X$ a set of all components of a correspondence Φ, and note that restriction of π to $\mathcal{K}_\Phi$ is a one-to-one mapping of $\mathcal{K}_\Phi$ onto $\mathcal{K}_{\Phi^{-1}}$, the inverse of which is restriction of π^{-1} to $\mathcal{K}_{\Phi^{-1}}$.

A component $H \in \mathcal{K}_\Phi$ is called *main*, if for some $x \in X$ we have $H = \overline{\{x\}} = \pi^{-1}(\pi(x))$. We say that H is generated by an element x, or x generates H. If the maximal component X is main, then the element that generates X is called a *unit*. A *canonical* mapping from X to $\mathcal{K}_\Phi$ transforms element x to a main component that is generated by x.

5 Examples: relations

Remind [2], that *relation* R on a set X is any subset $R \subseteq X \times X$ of cartesian product of X to itself. Thus relation is a special case of correspondence with $Y = X$. Let us discuss the sense of concepts defined above in this special case.

First state some properties of a correspondence as a relation R on a set X.

1. $R \subseteq X \times X$ is symmetric if and only if $R^{-1} = R$.

2. $R \subseteq X \times X$ is transitive if and only if $R \circ R \subseteq R$.

3. $R \subseteq X \times X$ is reflexive if and only if $R \supseteq I_X$.

4. $R \subseteq X \times X$ is antisymmetric if and only if $R^{-1} \cap R \subseteq I_X$.

5.1 Equivalence relation

Let $(X, \sim)$ be a set with equivalence relation, that is, symmetric, reflexive and transitive relation [2]. For convenience we will use both $\sim$ and R notation for this relation. Property 1 from section 5 implies that R^{-1} coincides with R, so inverse polar coincides wih direct one: $\pi = \pi^{-1}$. A value of polar π on a singleton $\{x\}$ is the equivalence class $K(x)$ containing x. If a set $A \subseteq X$ is contained in an equivalence class $C \in X/R$, then $\pi(A) = C$. If A contains at least two nonequivalent points then $\pi(A) = \emptyset$. Thus polar π values of R are empty set and equivalence classes.

Equivalence classes $C \in X/R$ are obviously components of R. For a trivial equivalence relation ($R = X \times X$, all elements of X are equivalent) X is the only equivalence class and both minimal and maximal component of R. This component is main and each element $x \in X$ may serve a unit. For a nontrivial equivalence relation R minimal component (center) is $\emptyset$, and maximal component is X, all other components coincide with equivalence classes and are main (any element of a class might be its generator).

A canonical mapping [2] from X to factor set X/R is the same as defined in section 4.

5.2 Order relation

Let $(X, \leq)$ be an ordered set, that is, a set with order relation [2]. For $x, y \in X$ define order segments

$$(\leftarrow, x] = \{z \in X |\ z \leq x\}, \quad [x, \rightarrow) = \{z \in X |\ x \leq z\}.$$

Clearly

$$\pi(x) = \Phi(x) = [x, \rightarrow), \quad \pi^{-1}(x) = \Phi^{-1}(x) = (\leftarrow, x], \quad x \in X.$$

Usual order on the set of reals possesses only components that are segments (denoted in this case by $(-\infty, x]$ $[x, \infty)$). Partial order may possess other components, see exercise 6.5.

6 Exercises

Exercise 6.1 *Derive a formula for inversion of a composition of correspondences, representing* $(\Phi \circ \Psi)^{-1}$ *via a composition of* Φ^{-1} *and* Ψ^{-1}*.*

Exercise 6.2 *Prove properties 1–8 in section 1.*

Exercise 6.3 *Prove properties 1–6 in section 2.*

Exercise 6.4 *Prove properties 1–4 in section 5*

Exercise 6.5 *Let* $X = \mathbf{R}^2$*, elements of* X *are represented by pairs* $x = (x_1, x_2)$*, and a partial order on* X *is defined by*

$$x \leq y \iff x_1 \leq y_1,\ x_2 \leq y_2.$$

Describe all components of this relation.

References

[1] Akilov G.P., Kutateladze S.S. (1978) *Ordered vector spaces.* Novosibirsk: Nauka, 368 p. (in Russian).

[2] Novosyolov A.A. (2002) Relations. *Lecture for KSU students* . Krasnoyarsk, 10 p., http://www.geocities.com/novosyolov/lectures.htm#Rel