KRASNOYARSK STATE UNIVERSITY

CHAIR OF APPLIED MATHEMATICS

Risk theory

Topic: Distorted probability measure

Lecture for students of math department of KSU

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Last modified 2/6/2002

Krasnoyarsk 2002

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Abstract

Distorted probability measure is being considered, its properties are studied, and methods of its calculation and statistical estimation are presented.

1 Introduction and notation

One of basic problems in risk theory is building a risk measure which is monotone with respect to a preference relation [1]. Here we consider a class of risk measures introduced in [2], [3], and study properties of the risk measures. This distorted probability measure was originally intended for insurance premium calculation, but may be used in a wider class of problems, including portfolio analysis.

Denote $\mathcal{X}$ the set of all real random variables, and $\mathcal{X}_+$ the set of nonnegative random variables:

$$\mathcal{X}_+ = \{X \in \mathcal{X} |\, \mathbf{P}\{X \geq 0\} = 1\}.$$

Introduce also specific notation for sets of random variables with finite expectation:

$$\widetilde{\mathcal{X}} = \{X \in \mathcal{X} :\ \mathbf{E}|X| < \infty\},\ \ \widetilde{\mathcal{X}}_+ = \{X \in \mathcal{X}_+ :\ \mathbf{E}|X| < \infty\}.$$

Let $F_X(x) = \mathbf{P}\{X \leq x\}$, $x \in \mathbf{R}$ denote the (cumulative) distribution function of a random variable $X \in \mathcal{X}$, and $S_X(x) = \mathbf{P}\{X > x\} = 1 - F_X(x)$, $x \in \mathbf{R}$ be its decumulative distribution function.

Let $g:\ [0,1]\to[0,1]$ be a nondecreasing function with $g(0)=0, g(1)=1$. Denote $\mathcal{G}$ the collection of all such functions. One can easily see that for each $g\in\mathcal{G}$ there is a dual function $\widetilde{g}\in\mathcal{G}$, defined by

$$\widetilde{g}(x)=1-g(1-x),\ x\in[0,1]. \tag{1}$$

It is clear that $\widetilde{\widetilde{g}}=g$.

Wang [2] introduced a distorted probability measure for nonnegative random variables

$$\pi_g(X)=\pi(X)=\int_0^\infty g(S_X(t))\,dt,\quad X\in\mathcal{X}_+, \tag{2}$$

and Young [3] modified (2) to get the measure for arbitrary random variable:

$$\pi_g(X)=\pi(X)=\int_{-\infty}^0 [g(S_X(t))-1]\,dt+\int_0^\infty g(S_X(t))\,dt,\quad X\in\mathcal{X}. \tag{3}$$

Note that risk measures (2) and (3) depend only on distribution function S_X of a random variable X.

Given a function $g\in\mathcal{G}$, denote $\mathcal{X}_g$ a class of random variables for which the value of risk measure (3) is finite:

$$\mathcal{X}_g=\{X\in\mathcal{X}|\ |\pi_g(X)|<\infty\}. \tag{4}$$

Next, let $\widetilde{\pi}$ be the dual risk measure defined by

$$\widetilde{\pi}_g(X)=\pi_{\widetilde{g}}(X),\quad X\in\mathcal{X}_{\widetilde{g}}. \tag{5}$$

2 Representation of expectation

In this section we will derive two useful representations for expectation of a random variable.

Lemma 2.1 *For $X\in\widetilde{\mathcal{X}}$ the following representation is true:*

$$\mathbf{E}X=-\int_{-\infty}^0 F_X(t)\,dt+\int_0^\infty (1-F_X(t))\,dt. \tag{6}$$

Proof. Indeed,

$$\mathbf{E}X = \int_{-\infty}^{\infty} t\,dF_X(t) = \int_{-\infty}^{0} t\,dF_X(t) + \int_{0}^{\infty} t\,dF_X(t). \tag{7}$$

Consider the case $|\mathbf{E}X| < \infty$, in particular, both integrals in (7) are finite (proof for the case $\mathbf{E}|X| = \infty$ is left to reader, see exercise 5.1). This implies that tails of integrals are infinitesimal:

$$\lim_{A\to-\infty} \int_{-\infty}^{A} t\,dF_X(t) = 0, \quad \lim_{B\to\infty} \int_{B}^{\infty} t\,dF_X(t) = 0. \tag{8}$$

Besides,

$$\int_{-\infty}^{A} t\,dF_X(t) \le A \int_{-\infty}^{A} dF_X(t) = AF(A) \le 0$$

and

$$\int_{B}^{\infty} t\,dF_X(t) \ge B \int_{B}^{\infty} dF_X(t) = B(1 - F(B)) \ge 0,$$

that together with (8) provides

$$\lim_{A\to-\infty} AF(A) = 0, \quad \lim_{B\to\infty} B(1 - F(B)) = 0.$$

Thus

$$\begin{aligned} \int_{-\infty}^{0} t\,dF_X(t) &= tF_X(t)|_{-\infty}^{0} - \int_{-\infty}^{0} F_X(t)\,dt \\ = -\lim_{t\to-\infty} tF_X(t) - \int_{-\infty}^{0} F_X(t)\,dt &= -\int_{-\infty}^{0} F_X(t)\,dt \end{aligned} \tag{9}$$

and

$$\begin{aligned} \int_{0}^{\infty} t\,dF_X(t) = -\int_{0}^{\infty} t\,d(1 - F_X(t)) &= -t(1 - F_X(t))|_{0}^{\infty} + \int_{0}^{\infty} (1 - F_X(t))\,dt \\ = -\lim_{t\to\infty} t(1 - F_X(t)) + \int_{0}^{\infty} (1 - F_X(t))\,dt &= \int_{0}^{\infty} (1 - F_X(t))\,dt \end{aligned} \tag{10}$$

Substitution of (9), (10) into (7) implies the lemma. ◇

Lemma 2.2 *For $X \in \widetilde{\mathcal{X}}$ the following representation is true*

$$\mathbf{E}X = \int_{0}^{1} S_X^{-1}(v)\,dv. \tag{11}$$

Proof. Changing variables $v = S_X(t)$ and integrating by parts gives

$$-\int_{-\infty}^{0} F_X(t)\,dt = \int_{-\infty}^{0} (S_X(t)-1)\,dt = \int_{1}^{S(0)} (v-1)\,dS_X^{-1}(v)$$
$$= (v-1)S_X^{-1}(v)|_1^{S(0)} - \int_1^{S(0)} S_X^{-1}(v)\,dv = \int_{S(0)}^{1} S_X^{-1}(v)\,dv \tag{12}$$

and

$$\int_0^{\infty} (1-F_X(t))\,dt = \int_0^{\infty} S_X(t)\,dt = \int_{S(0)}^{0} v\,dS_X^{-1}(v)$$
$$= vS_X^{-1}(v)|_{S(0)}^{0} - \int_{S(0)}^{0} S_X^{-1}(v)\,dv = \int_0^{S(0)} S_X^{-1}(v)\,dv. \tag{13}$$

Substitution of (12) and (13) into (6) completes the proof. ⋄

3 Properties of distorted probability measure

Proposition 3.1 *If $g(t) \equiv t$, then $\pi(X) = \mathbf{E}X$.*

Proof. Indeed, condition of lemma implies that risk measure π has the form

$$\pi(X) = \int_{-\infty}^{0} (S_X(t)-1)\,dt + \int_0^{\infty} S_X(t)\,dt,$$

that by lemma 2.1 coincides with $\mathbf{E}X$. ⋄

Let us now study conditions for existence of integral in (3).

Proposition 3.2 *$\widetilde{\mathcal{X}} = \mathcal{X}_g$ if and only if*

$$0 < g'(0) < \infty, \;\; 0 < g'(1) < \infty. \tag{14}$$

Proof. Let (14) be true. Show that $X \in \widetilde{\mathcal{X}}$ implies $X \in \mathcal{X}_g$. Indeed, $g'(0) < \infty$ implies existence of $0 < M < \infty$ and $\delta \in (0,1)$ such that $g(x) \le Mx$ for $x \in [0,\delta]$. Choose $B \in (0,\infty)$ such that $S_X(t) \le \delta$ for $t \ge B$. Now

$$\int_0^{\infty} g(S_X(t))\,dt = \int_0^{B} g(S_X(t))\,dt + \int_B^{\infty} g(S_X(t))\,dt \le B + M\int_B^{\infty} S_X(t)\,dt.$$

Since $\mathbf{E}X$ exists, the integral in the right hand side is finite, hence

$$0 \le \int_0^\infty g(S_X(t))\,dt < \infty. \tag{15}$$

Next, $g'(1) < \infty$ implies existence of $0 < M < \infty$ and $\delta \in (0,1)$ such that $g(x) \ge 1 - Mx$ for $x \in [\delta, 1]$. Choose $A \in (-\infty, 0)$ such that $S_X(t) \ge \delta$ for $t \le A$. Then

$$\int_{-\infty}^0 [g(S_X(t))-1]\,dt = \int_{-\infty}^A [g(S_X(t))-1]\,dt + \int_A^0 [g(S_X(t))-1]\,dt \ge -M\int_{-\infty}^A S_X(t)\,dt + A.$$

Here integral in the right hand side is also finite due to existence of $\mathbf{E}X$, thus

$$0 \ge \int_{-\infty}^0 [g(S_X(t)) - 1]\,dt > -\infty. \tag{16}$$

Using (15), (16) in (3), we get $|\pi(X)| < \infty$, that is $X \in \mathcal{X}_g$. Implication $X \in \mathcal{X}_g \Longrightarrow X \in \widetilde{\mathcal{X}}$ follows similarly from $g'(0) > 0$ and $g'(1) > 0$.

We have shown that (14) implies coincidence of $\widetilde{\mathcal{X}}$ and $\mathcal{X}_g$. To illustrate the inverse we will present examples of distributions that show difference of the classes when conditions in (14) are broken.

Let $g(x) = x^2,\ x \in [0,1]$. Clearly $g'(0) = 0$. Consider a random variable X with decumulative distribution function

$$S_X(t) = \begin{cases} (1+t)^{-1}, & t \ge 0, \\ 0, & t < 0. \end{cases}$$

We have

$$\pi(X) = \int_0^\infty S_X^2(t)\,dt = 1,$$

while expectation of X is obviously infinite. So $X \in \mathcal{X}_g,\ X \notin \widetilde{\mathcal{X}}$. Other variants of differences between $\widetilde{\mathcal{X}}$ and $\mathcal{X}_g$ are left to reader (see exercises 5.2, 5.3). $\diamond$

Given $b \in \mathbf{R}$, denote W_b the degenerate random variable: $\mathbf{P}\{W_b = b\} = 1$, and consider some elementary properties of π.

Proposition 3.3 *Risk measure π possesses the following properties:*

1. $\pi(W_b) = b,\ b \in \mathbf{R}$;

2. $\pi(X + b) = \pi(X) + b,\ b \in \mathbf{R},\ X \in \mathcal{X}_g$;

3. $\pi(aX) = a\pi(X),\ a \geq 0,\ X \in \mathcal{X}_g$;

4. $\pi_g(aX) = a\pi_{\widetilde{g}}(X),\ a < 0,\ X \in \mathcal{X}_g$, *where* $\widetilde{g}(x) = 1 - g(1 - x),\ x \in [0, 1]$.

Proof. 1. Decumulative distribution function of W_b has the form

$$S_{W_b}(t) = \begin{cases} 1, & t < b, \\ 0, & t \geq b. \end{cases}$$

For $b \geq 0$ (3) implies

$$\pi(W_b) = \int_0^b g(1)\, dt = b,$$

and for $b < 0$ one gets:

$$\pi(W_b) = \int_b^0 [g(0) - 1]\, dt = -\int_b^0 dt = b.$$

2. Noticing that $S_{X+b}(t) = S_X(t - b),\ t \in \mathbf{R}$ and changing variables by $u = t - b$ we get from (3) a representation

$$\pi(X + b) = \int_{-\infty}^{-b} [g(S_X(u)) - 1]\, du + \int_{-b}^{\infty} g(S_X(u))\, du.$$

This directly leads to

$$\begin{aligned} \pi(X + b) = \pi(X) - \int_{-b}^0 [g(S_X(u)) - 1]\, du + \int_{-b}^0 g(S_X(u))\, du \\ = \pi(X) + \int_{-b}^0 du = \pi(X) + b, \end{aligned}$$

as required.

3. Equality is trivial for $a = 0$. For $a > 0$ we have $S_{aX}(t) = S_X(t/a),\ t \in \mathbf{R}$, thus changing variables: $u = t/a$, we get

$$\begin{aligned} \pi(aX) = \int_{-\infty}^0 [g(S_{aX}(t)) - 1]\, dt + \int_0^{\infty} g(S_{aX}(t))\, dt \\ = a\int_{-\infty}^0 [g(S_X(u)) - 1]\, du + a\int_0^{\infty} g(S_X(u))\, du = a\pi(X), \end{aligned}$$

as required.

4. Let $a < 0$. For almost all $t \in \mathbf{R}$ we have

$$S_{aX}(t) = \mathbf{P}\{aX > t\} = \mathbf{P}\{X < t/a\} = 1 - S_X(t/a).$$

Changing variables $u = t/a$ once more, one gets

$$\begin{aligned}\pi_g(aX) &= \int_{-\infty}^{0} [g(S_{aX}(t)) - 1]\, dt + \int_0^\infty g(S_{aX}(t))\, dt \\ &= \int_{-\infty}^{0} [g(1 - S_X(t/a)) - 1]\, dt + \int_0^\infty g(1 - S_X(t))\, dt \\ &= -a \int_\infty^0 \widetilde{g}(S_X(u)\, du - a \int_0^{-\infty} [\widetilde{g}(S_X(u)) - 1]\, du \\ &= a \int_{-\infty}^{0} [\widetilde{g}(S_X(u)) - 1]\, du + a \int_0^\infty \widetilde{g}(S_X(u))\, du = a\pi_{\widetilde{g}}(X).\end{aligned}$$

The proof is complete. $\diamond$

Proposition 3.4 *The distorted probability measure may be written in the form*

$$\pi_g(X) = \int_0^1 S_X^{-1}(v)\, dg(v). \tag{17}$$

Proof is left to reader (exercise 5.4).

Proposition 3.5 *For any $g \in \mathcal{G}$, $X \in \mathcal{X}_g$ we have*

$$\pi_g(-X) = -\widetilde{\pi}_g(X),$$

where $\widetilde{\pi}_g$ is a dual risk measure defined in (5).

Proof. For any distribution and almost all $t \in \mathbf{R}$ we have

$$S_{-X}(t) = \mathbf{P}\{-X > t\} = \mathbf{P}\{X < -t\} = 1 - S_X(-t). \tag{18}$$

From (3) by elementary transformations, using (1) and (18), we obtain

$$\begin{aligned}\pi_g(-X) &= \int_{-\infty}^{0} [g(S_{-X}(t)) - 1]\, dt + \int_0^\infty g(S_{-X}(t))\, dt \\ &= \int_{-\infty}^{0} [g(1 - S_X(-t)) - 1]\, dt + \int_0^\infty g(1 - S_X(-t))\, dt \\ &= -\int_{-\infty}^{0} [\widetilde{g}(S_X(-t))]\, dt + \int_0^\infty [1 - \widetilde{g}(S_X(-t))].\, dt\end{aligned}$$

Changing variables $u = -t$ gives

$$\begin{aligned}\pi_g(-X) &= \int_{\infty}^{0} \widetilde{g}(S_X(u))\,du + \int_{0}^{-\infty} [\widetilde{g}(S_X(u)) - 1]\,du \\ &= -\int_{-\infty}^{0} [\widetilde{g}(S_X(u)) - 1]\,du - \int_{0}^{\infty} \widetilde{g}(S_X(u))\,du \\ &= -\pi_{\widetilde{g}}(X) = -\widetilde{\pi}_g(X),\end{aligned}$$

as needed. $\diamond$

Corollary 3.1 *If X is symmetric then*

$$\pi_g(X) + \widetilde{\pi}_g(X) = 0.$$

Proof is a direct consequence of proposition 3.5, since X and $-X$ have the same distribution. $\diamond$

4 Calculation and estimation

Here we will consider a method for calculation of distorted probability measure for a discrete distribution and a method of its statistical estimation from observations.

Let the distribution of X be restricted to a finite number of points:

$$\mathbf{P}\{X = x_k\} = p_k,\ k = 1, \ldots, n;\ \ p_1 \geq 0, \ldots, p_n \geq 0,\ \ p_1 + \ldots + p_n = 1. \tag{19}$$

Without loss of generality we may assume that the sequence $x_k,\ k = 1, \ldots, n$ is increasing: $x_1 < x_2 < \ldots < x_n$.

Proposition 4.1 *Distorted probability measure for a distribution (19) may be calculated by*

$$\pi_g(X) = \sum_{s=1}^{n} g\left(\sum_{k=s}^{n} p_k\right)(x_s - x_{s-1}), \tag{20}$$

where $x_0 = 0$, or by

$$\pi_g(X) = \sum_{s=1}^{n} x_s \left[g\left(\sum_{k=s}^{n} p_k\right) - g\left(\sum_{k=s+1}^{n} p_k\right)\right], \tag{21}$$

where empty sum $\sum_{k=n+1}^{n} p_k$ is equal zero.

Proof. First derive (21) from (17). Note that

$$S_X^{-1}(v) = x_k, \quad v \in \left(\sum_{s=k+1}^{n} p_k, \sum_{s=k}^{n} p_k \right), \quad k = 1, \ldots, n.$$

Nonuniqueness of S_X^{-1} at $\sum p_k$ does not effect the value of integral in (17), thus values of the function S_X^{-1} at these points may be chosen arbitrarily. Now:

$$\begin{aligned} \pi_g(X) &= x_n g(p_n) + x_{n-1}(g(p_n + p_{n-1}) - g(p_n)) \\ &\qquad + \ldots + x_1(g(1) - g(p_n + \ldots + p_2)) \\ &= \sum_{s=1}^{n} x_s \left[g\left(\sum_{k=s}^{n} p_k\right) - g\left(\sum_{k=s+1}^{n} p_k\right)\right], \end{aligned}$$

as required.

Equivalence of (20) and (21) is shown using "summation by parts" as presented in lemma 4.1. ◇

Lemma 4.1 *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be any real numbers. Then*

$$\sum_{s=1}^{n} a_s(b_s - b_{s-1}) = \sum_{s=1}^{n} b_s(a_s - a_{s+1}),$$

where $a_0 = b_{n+1} = 0$.

Proof: see exercise 5.6.

Now consider a simple method of statistical estimation of $\pi_g(X)$ from observations. Let $X_1, ..., X_n$ be a sequence of independent identically distributed random variables with the same distribution as X, and let $X_{(1)}, \ldots, X_{(n)}$ be the corresponding order statistics (that is, ordered samples). Using empirical distribution, that prescribes probabilities $1/n$ to sample points $X_1, \ldots, X_n$, in (20), (21), one obtains

$$\widehat{\pi}_g(X) = \sum_{s=1}^{n} g\left(\frac{n-s+1}{n}\right)(X_{(s)} - X_{(s-1)}) \tag{22}$$

$$= \sum_{s=1}^{n} X_{(s)} \left[g\left(\frac{n-s+1}{n}\right) - g\left(\frac{n-s}{n}\right)\right]. \tag{23}$$

5 Exercises

Exercise 5.1 *Prove lemma 2.1 for the case* $\mathbf{E}|X| = \infty$.

Exercise 5.2 *Let* $g(x) = \sqrt{x}$, $x \in [0,1]$, *and a random variable* X *has a cumulative distribution function*

$$S_X(t) = \begin{cases} (1+t)^{-2}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Show that $X \in \widetilde{\mathcal{X}}$ *but* $X \notin \mathcal{X}_g$.

Exercise 5.3 *Construct examples of differences of classes* $\widetilde{\mathcal{X}}$ *and* $\mathcal{X}_g$ *when conditions* $g'(1) > 0$ *and* $g'(1) < \infty$ *are not satisfied. (*Hint*. To construct examples it is necessary to choose* $S_X(t)$ *with specific behavior as* $t \to -\infty$*).*

Exercise 5.4 *Prove proposition 3.4 using the same method as in lemma 2.2.*

Exercise 5.5 *Derive formulae for calculation of distorted probability measure for discrete distributions restricted to the set of nonnegative integers* $\mathcal{N} = \{0, 1, 2, \ldots\}$ *and the set of all integers* $\mathcal{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Exercise 5.6 *Prove the summation by parts lemma 4.1) and show equivalence of (20) and (21).*

References

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