Krasnoyarsk state university

Chair of applied mathematics

Risk theory

Topic: Investment portfolio selection

Lecture for students of math department of KSU

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1 Introduction

A typical problem of risk theory is that of optimal distribution of restricted resourses, say, investment capital. Examples are selection of investment portfolio or geographical distribution of production.

Let $(X_1, ..., X_n)$ be a fixed random vector, $m = (m_1, ..., m_n)$ be the vector of its mean values, $V = (v_{ij})$ be its covariance matrix, that is

$$m_i = \mathbf{E}X_i, \;\; v_{ij} = \mathbf{E}(X_i - m_i)(X_j - m_j), \;\; i, j = 1, ..., n. \tag{1}$$

Let further $y = (y_1, ..., y_n)$ be a vector containing in

$$L_n = \{y \in \mathbf{R}^n |\; y_1 + ... + y_n = 1\}. \tag{2}$$

Random variable

$$\mathcal{P} = \mathcal{P}(y) = y_1 X_1 + ... + y_n X_n, \tag{3}$$

is called portfolio and components of y are called weights of tools X_i, $i = 1, ..., n$ in the portfolio. Expectation and variance of portfolio (3), are equal to (see exercise 5.1)

$$\mathbf{E}\mathcal{P} = y_1 m_1 + ... + y_n m_n = y^T m, \ \ \mathbf{D}\mathcal{P} = \sum_{i=1}^{N} \sum_{j=1}^{N} v_{ij} y_i y_j = y^T V y, \tag{4}$$

where super index T denotes transpose.

Portfolio selection problem consists in choosing the best in some sense weights y. Standard way of comparing random variables (more precisely - their distributions) is defining a risk measure μ, that is a mapping from a set of random variables $\mathcal{X}$ (or a set of distribution functions $\mathcal{F}$) into reals:

$$\mu : \mathcal{X} \to \mathbf{R}, \ \ (\mu : \mathcal{F} \to \mathbf{R}).$$

Risk measure of a portfolio $\mathcal{P}$ turns out to be a function of weights y:

$$\mu(\mathcal{P}) = \mu(y_1 X_1 + ... + y_n X_n) = f(y),$$

thus portfolio selection problem takes the form

$$f(y) \to \max_y (\min_y) \tag{5}$$

subject to

$$y \in L_n \tag{6}$$

and perhaps additional restrictions related to specific problems.

Denote $I = (1, 1, ..., 1)^T$, then the set L_n is described by

$$L_n = \{ y \in \mathbf{R}^n | \ y^T I = 1 \}. \tag{7}$$

In what follows we will assume that covariance matrix V is nondegenerate, thus both V and its inverse V^{-1} are symmetric positively defined. This allows defining in $\mathbf{R}^n$ an inner product of the form

$$(u, v) = u^T V^{-1} v, \quad u, v \in \mathbf{R}^n \tag{8}$$

and related norm $\|u\| = \sqrt{(u, u)}$, which is sometimes called energy norm. In some problems weights should be nonnegative¡ so optimization problem (5) is stated not on the whole L_n, but rather on its subset

$$\mathbf{S}_n = \{y \in \text{Ł}_n |\ y_1 \geq 0, ..., y_n \geq 0\}, \tag{9}$$

a standard simplex in $\mathbf{R}^n$.

2 A simple portfolio

Let $\mathcal{X}$ be the set of random variables having finite second moment: $\mathbf{E}X^2 < \infty$. Variance may be chosen as a risk measure on this set:

$$\mu(X) = \mathbf{D}X,\ X \in \mathcal{X}. \tag{10}$$

Denoting D_F variance of a random variable possessing distribution function F, we may consider this risk measure on the set $\mathcal{F}$ of distributions with finite second moment:

$$\mu(F) = D_F,\ F \in \mathcal{F}. \tag{11}$$

Variance is to be minimized, so the problem (5), (6) reduces to

$$y^T V y \to \min_y, \tag{12}$$

$$y^T I = 1. \tag{13}$$

Now apply Lagrange multipliers method to the above problem. Lagrange function takes the form

$$\mathcal{L}(y, \lambda) = y^T V y + \lambda(y^T I - 1),$$

so its partial derivatives are

$$\mathcal{L}_y = 2Vy + \lambda I,$$

$$\mathcal{L}_\lambda = y^T I - 1.$$

Equating them to 0, one obtains from the first equation

$$y = -\frac{1}{2}\lambda V^{-1} I.$$

Now substituting y into second equation we get $\lambda = -2/\|I\|^2$, so vector

$$y^* = \frac{V^{-1}I}{\|I\|^2}. \tag{14}$$

is the solution of (12), (13). Remark that expected return and variance of the optimal portfolio are equal to

$$\mathbf{E}\mathcal{P}(y^*) = y^{*T}m = \frac{(m, I)}{\|I\|^2}, \quad f(y^*) = \frac{1}{\|I\|^2}. \tag{15}$$

To illustrate let us write down the solution for uncorrelated tools (components of X). Covariance matrix V is diagonal in this case: $V = diag(\sigma_1^2, ..., \sigma_n^2)$, so $V^{-1} = diag(\sigma_1^{-2}, ..., \sigma_n^{-2})$, $\|I\|^2 = \sum_{i=1}^n \sigma_i^{-2}$ and

$$y_i^* = \frac{\sigma_i^{-2}}{\sigma_1^{-2} + ... + \sigma_n^{-2}}, \quad i = 1, 2, ..., n, \quad f(y^*) = \frac{1}{\sigma_1^{-2} + ... + \sigma_n^{-2}}.$$

In particular, if $\sigma_1 = ... = \sigma_n = \sigma$, then

$$y_i^* = \frac{1}{n}, \; i = 1, ..., n, \quad f(y^*) = \frac{\sigma^2}{n}.$$

3 Reconciling risk and return

The risk and return paradigm suggests that investor is seeking for maximal return of a portfolio (that was denoted by $\mathbf{E}\mathcal{P}$), and simultaneously tries to minimize "risk" $\mathbf{D}\mathcal{P}$. In what follows we consider general statement of such vector optimization problems and point to two approaches for their solution.

3.1 Vector optimization

Let $g_1, ..., g_m$ be functions on a set $D \in \mathbf{R}^n$, taking values in $\mathbf{R}$. The problem

$$g_1(y) \to \min_y,$$

$$...$$

$$g_m(y) \to \min_y,$$

$$y \in D$$

is called vector (multicriteria) optimization problem. Since points y in which minima of different functions g_i, $i = 1, ..., m$ are achieved are usually different, the very concept of solution needs definition. The following is an outline of two approaches to the problem.

First approach may be described as follows: fix values of all goal functions except for one of them, say, the first, and then solve usual restricted optimization problem:

$$g_1(y) \to \min_y,$$

$$g_2(y) = a_2,$$

$$...$$

$$g_m(y) = a_m,$$

$$y \in D,$$

with subsequent studying the dependence of solution on parameters $a_2, ..., a_m$.

Another approach consists in assigning positive weights $\alpha_1, ..., \alpha_m$ to goal functions (one of weights, say α_m may always be set to 1) and solving an optimization problem with weighted criterion

$$g(y) = \alpha_1 g_1(y) + ... + \alpha_{m-1} g_{m-1}(y) + g_m(y) \to \min_y,$$

$$y \in D.$$

Consider both approaches applied to optimal portfolio selection in the sence of expected return and variance: $g_1(y) = y^T V y$, $g_2(y) = -y^T m$, $y \in D = L_n$.

3.2 Markowitz problem

In [3] the first of approaches mentioned was used: expected return $y^T m$ of a portfolio is set to a fixed value M and portfolio variance $y^T V y$ is being minimized. This leads to a problem similar to (12)–(13), with an additional restriction

$$y^T V y \to \min_y, \tag{16}$$

$$y^T I = 1, \tag{17}$$

$$y^T m = M, \tag{18}$$

where M is a parameter. Solving (16)–(18) by Lagrange multipliers method (see exercise 5.2), one obtains:

$$y^*(M) = uM + v, \tag{19}$$

where

$$u = V^{-1}\,\frac{m\|I\|^2 - I(m,I)}{\Delta}, \quad v = V^{-1}\,\frac{I\|m\|^2 - m(m,I)}{\Delta}$$

and

$$\Delta = \|I\|^2\|m\|^2 - (m,I)^2.$$

Note that dependence of solution y^* on parameter M is linear. Calculating portfolio variance with optimal weights (19), we get

$$\sigma^2 = f(y^*(M)) = \mathbf{D}\mathcal{P}(y^*(M)) = \frac{\|I\|^2 M^2 - 2(m,I)M + \|m\|^2}{\Delta}, \tag{20}$$

thus dependence of optimal "risk" on expected return is quadratic. Minimal value of variance in (20) is equal to $1/\|I\|^2$ and is reached with $M^* = (m,I)/\|I\|^2$ (see exercise 5.3). Note that optimal weights y^* corresponding to M^* are $y^*(M^*) = V^{-1}I/\|I\|^2$, the same as solution of (12)–(13). Note also that $I^T u = 0$, $I^T v = 1$, which means that vector v belongs to a hyperplane L_n (7), and vector u is parallel to the hyperplane.

The set of points on a plane (σ^2, M), satisfying (20) with $M \geq M^*$, is called efficient frontier [4]. Points of the set correspond to portfolios with minimal variance σ^2 given expected return M, or, equivalently, maximal expected return M given variance σ^2. Note that σ^2 is strictly increasing function of M on efficient frontier: "the more return, the more risk".

It is worth studying what's that M^*, a return, corresponding to a total minimum of variance. Consider a partial case $V = E$, the identity matrix (tools are uncorrelated and possess unit variance each). We have $V^{-1} = E$, so $\|I\|^2 = n$, $(m, I) = m_1 + \ldots + m_n$, and

$$M^* = \frac{m_1 + \ldots + m_n}{n}$$

is an average of tools' returns.

Now let V be diagonal, that is, tools are still uncorrelated, yet having different variances σ_i^2, $i = 1, \ldots, n$. Matrix V^{-1} is also diagonal with elements σ_i^{-2}, $i = 1, \ldots, n$, so

$$M^* = \frac{\sum_{i=1}^n \sigma_i^{-2} m_i}{\sum_{i=1}^n \sigma_i^{-2}},$$

thus M^* is a convex combination of $m_1, \ldots, m_n$ with coefficients

$$\frac{\sigma_k^{-2}}{\sum_{i=1}^n \sigma_i^{-2}} > 0, \quad k = 1, \ldots, n.$$

In particular, $\min\{m_1, \ldots, m_n\} \leq M^* \leq \max\{m_1, \ldots, m_n\}$.

One may suggest that M^* is always weighted average or convex combination of tools' returns. The first is actually true, while the second is not, since weights may be negative, thus leading to a nonconvex combination. In particular it is possible that

$$M^* < \min\{m_1, \ldots, m_n\}, \text{ or } M^* > \max\{m_1, \ldots, m_n\}.$$

Consider an example. Let $n = 3$,

$$V = \begin{pmatrix} 4 & 3/2 & 0 \\ 3/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then $y^*(M^*) = V^{-1}I/\|I\|^2 = (-2/15, 2/3, 7/15)^T$, so if $m = (1, 2, 2.1)^T$ then one gets $M^* = 2.18 > \max\{m_1, m_2, m_3\} = 2.1$, while setting $m = (2, 1, 1)^T$ leads to $M^* = (13/15) < \min\{m_1, m_2, m_3\} = 1$.

3.3 Weighted criterion

Now consider simultaneous optimization of expected return and variance by converting vector criterion to a scalar one. Let $\alpha > 0$ be a weighting parameter, consider the problem

$$g(y) = \frac{1}{2}\alpha y^T V y - y^T m \to \min_y, \tag{21}$$

$$y^T I = 1. \tag{22}$$

Solving it by Lagrange multipliers method (see exercise 5.4) leads to

$$y^* = y^*(\alpha) = \frac{1}{\alpha} V^{-1} \left[m - \frac{(m, I)}{\|I\|^2} I \right] + \frac{V^{-1} I}{\|I\|^2} \tag{23}$$

$$g(y^*) = \frac{\alpha - (m, I)}{\|I\|^2}, \tag{24}$$

with expected return and variance of the optimal portfolio expressed by

$$\mathbf{E}\mathcal{P}(y^*) = y^{*T} m = \frac{\Delta}{\alpha \|I\|^2} + \frac{(m, I)}{\|I\|^2}, \tag{25}$$

$$\mathbf{D}\mathcal{P}(y^*) = y^{*T} V y^* = \frac{\Delta}{\alpha^2 \|I\|^2} + \frac{1}{\|I\|^2}. \tag{26}$$

Consider behavior of solution when parameter α changes within $(0, \infty)$. From (25) and (26) it is clear that expected return and variance of the optimal portfolio are strictly decreasing functions of α, and

$$\lim_{\alpha \to \infty} y^*(\alpha) = \frac{V^{-1} I}{\|I\|^2}, \quad \lim_{\alpha \to \infty} \mathbf{E}\mathcal{P}(y^*(\alpha)) = \frac{(m, I)}{\|I\|^2}, \quad \lim_{\alpha \to \infty} \mathbf{D}\mathcal{P}(y^*(\alpha)) = \frac{1}{\|I\|^2},$$

that is, limiting solution coincides with (14), (15).

3.4 Attitude to risk

Parameter M of the Markowitz problem (16)–(18) may serve an indicator on investor's attutide to risk: the more risky investor would choose larger expected return M thus taking larger "risk" σ^2. Parameter α plays the similar role in (21)–(22): the less its value, the greater values of expected return and variance. Actually these parameters are connected even more deeply. To be precise: solving problems (16)–(18) and (21)–(22) with parameters M and α, satisfying relation

$$M = M(\alpha) = \frac{\Delta}{\alpha\|I\|^2} + \frac{(m,I)}{\|I\|^2}, \quad \alpha > 0, \tag{27}$$

or inverse relation

$$\alpha = \alpha(M) = \frac{\Delta}{M\|I\|^2 - (m,I)}, \quad M > M^* = \frac{(m,I)}{\|I\|^2}, \tag{28}$$

would lead to equal portfolios (see exercise 5.5).

4 Expected utility method

4.1 Statement of a problem

Consider quite different approach to portfolio selection based on utility theory [5]. Let $U: \mathbf{R} \to \mathbf{R}$ be a nondecreasing concave function, define a risk measure

$$\rho(X) = \mathbf{E}U(X), \quad X \in \mathcal{X}, \tag{29}$$

called expected utility of a risk X.

Expected utility of portfolio (3) is equal to

$$g(y) = \mathbf{E}U(\mathcal{P}) = \mathbf{E}U\left(y_1X_1 + ... + y_nX_n\right).$$

Consider optimization problem

$$g(y) \to \max_y, \tag{30}$$

$$y \in L_n. \tag{31}$$

Fucntional ρ is concave in value, so g is concave as a function of y on L_n [6], thus (30), (31) is a convex programming problem; a number of numeric methods do exist to solve the problem.

Consider a partial case and compare results with those obtained before.

4.2 Exponential utility and attitude to risk

Pratt [7] introduced a concept of risk price $\pi(X)$ as a solution of equation

$$U(\mathbf{E}X - \pi(X)) = \mathbf{E}U(X),$$

and defined risk aversion for smooth utility functions U:

$$a(x) = -\frac{U''(x)}{U'(x)}, \quad x \in \mathbf{R},$$

He also showed that more risky investor (having smaller risk aversion function) would take any risk for smaller price. Precisely, if two investors possess utility functions U_1, U_2 with corresponding risk aversion functions a_1, a_2 and risk prices π_1, π_2, then

$$a_1(x) \le a_2(x), \ x \in \mathbf{R} \Longleftrightarrow \pi_1(X) \le \pi_2(X), \ X \in \mathcal{X}.$$

By direct calculation one can easily see that for an exponential utility function

$$U(x) = 1 - \exp(-\alpha x), \ x \in \mathbf{R}, \tag{32}$$

where $\alpha > 0$ is a parameter, risk aversion is constant:

$$a(x) = \alpha, \quad x \in \mathbf{R}. \tag{33}$$

4.3 Normal distribution and expected utility

Let the joint distribution of tools $X = (X_1, ..., X_n)$ be normal. Then [8] portfolio distribution $\mathcal{P}(y)$ is also normal with parameters $\nu = y^T m$ and $\sigma^2 = y^T V y$ and density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\nu)^2}{2\sigma^2}\right). \tag{34}$$

Next suppose that utility function U is exponential as in (32). Then expected utility of a portfolio may be calculated in closed form:

$$\begin{aligned}\mathbf{E}U(\mathcal{P}(y)) &= \int_{-\infty}^{\infty} U(x) f(x)\,dx = \int_{-\infty}^{\infty} (1 - \exp(-\alpha x) f(x)\,dx \\ &= 1 - \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp(-\alpha x) \exp\left(-\frac{(x-\nu)^2}{2\sigma^2}\right)\,dx.\end{aligned}$$

Since integrand is equal to (see exercise 5.6)

$$\exp\left(-\frac{(x-(\nu-\alpha\sigma^2))^2}{2\sigma^2}\right)\exp\left(\frac{1}{2}\alpha^2\sigma^2-\alpha\nu\right), \tag{35}$$

we get

$$\begin{aligned}\mathbf{E}U(\mathcal{P}(y)) &= 1-\exp\left(\frac{1}{2}\alpha^2\sigma^2-\alpha\nu\right)\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-(\nu-\alpha\sigma^2))^2}{2\sigma^2}\right)dx \\ &= 1-\exp\left(\frac{1}{2}\alpha^2\sigma^2-\alpha\nu\right) = 1-\exp\left(\frac{1}{2}\alpha^2 y^T V y-\alpha y^T m\right),\end{aligned}$$

so maximization of $\mathbf{E}\mathcal{P}(y)$ is equivalent to minimization of

$$g(y) = \frac{1}{2}\alpha y^T V y - y^T m$$

with restriction $y \in \text{Ł}_n$, that is the same as (21)–(22). Thus maximizing exponential expected utility under normal distribution of tools provides the same portfolio as weighted criterion method. Note that risk aversion parameter α has the same sence as parameter of the problem (21)–(22).

5 Exercises

Exercise 5.1 *Calculate mean and variance (4) of a portfolio (3) given characteristics m, V of investment tools.*

Exercise 5.2 *Solve the Markowitz problem (16)–(18) by Lagrange multipliers method.*

Exercise 5.3 *Calculate minimum point and minimal variance on efficient frontier of the Markowits problem (20).*

Exercise 5.4 *Solve the problem (21), (22) by Lagrange multipliers method. Calculate expected return (25) and variance (26) of the optimal portfolio.*

Exercise 5.5 *Prove that if parameters M and α in the problems (16)–(18) and (21)–(22), respectively, are related as in (27), (28), then the problems possess identical solutions.*

Exercise 5.6 *Prove the formula (35).*

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