Krasnoyarsk State University

Chair of Applied Mathematics

Risk Theory

Topic: Relations

Lecture for math department students

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Abstract

A concept of relation on a fixed set is introduced. Equivalence, order and preference relations are considered and some interrelations among these concepts are established.

1 Introduction

Risk theory as a decision theory under probabilistic uncertainty is based on a concept of preference on a set of probability distributions. In this lecture we introduce a concept of relation and consider preference as well as equivalence and order relations. This material is used in risk theory to establish connection of preference on a set of probability distributions with functionals defined on this set.

2 Relation

2.1 Definition

Let $\mathcal{X}$ be a fixed set.

Definition 2.1 *A relation Q on a set $\mathcal{X}$ is any subset of cartesian product $Q \subseteq \mathcal{X} \times \mathcal{X}$.*

We will say that there is a relation Q between elements $x, y \in \mathcal{X}$ if $(x, y) \in Q$; sometimes this is also written as xQy.

2.2 Examples

Example 2.1 *Consider a subset $I_{\mathcal{X}} = \{(x, x)|\ x \in \mathcal{X}\} \subseteq \mathcal{X} \times \mathcal{X}$. This subset defines an* equality *relation. For $\mathcal{X} = \{x, y, z\}$ consisting of 3 elements this relation is shown in the following figure, where elements of $\mathcal{X} \times \mathcal{X}$, belonging to $I_{\mathcal{X}}$, are shown as circles.*

	x	y	z
x	○	·	·
y	·	○	·
z	·	·	○

Example 2.2 *Next figure shows a relation on $\mathcal{X} = \{x, y, z, t\}$ that does not have any specific name, being anyway a relation in the sence of definition 2.1.*

	x	y	z	t
x	·	○	·	·
y	○	·	·	·
z	○	·	·	·
t	·	○	·	·

2.3 Relations properties

Now consider properties which will be used in axiomatic description of relations.

- Completeness. A relation Q on a set $\mathcal{X}$ is called complete, if for any pair of elements $x, y \in \mathcal{X}$ at least one pair (x, y), (y, x) belongs to Q.
- Symmetry. A relation Q on a set $\mathcal{X}$ is called symmetric, if $(x, y) \in Q$ implies $(y, x) \in Q$.
- Transitivity. A relation Q on a set $\mathcal{X}$ is called transitive, if $(x, y) \in Q$ and $(y, z) \in Q$ imply $(x, z) \in Q$.
- Antisymmetry. A relation Q on a set $\mathcal{X}$ is called antisymmetric, if $(x, y) \in Q$ and $(y, x) \in Q$ imply $x = y$.
- Reflexiveness. A relation Q on a set $\mathcal{X}$ is called reflexive, if $(x, x) \in Q,\ x \in \mathcal{X}$; in other words $I_{\mathcal{X}} \subseteq Q$.

It is worth pointing out that complete relation is necessarily reflexive (see exercise 5.1), but no other interconnections between these properties exist (exercise 5.2).

3 Specific relations

Here some commonly used relations will be introduced.

3.1 Equivalence

3.1.1 Definition

Definition 3.1 *Reflexive, symmetric and transitive relation Q on a set $\mathcal{X}$ is called* equivalence *relation.*

This relation if often denoted by the special sign: $(x, y) \in Q$ is written as $x \sim y$. An example of equivalence is equality on $\mathcal{X}$:

$$x \sim y \iff x = y.$$

Since equivalence Q is reflexive by definition, we always have $I_{\mathcal{X}} \subseteq Q$, thus equality being the least (with respect to inclusion) equivalence relation on $\mathcal{X}$.

Another example is total relation $T_{\mathcal{X}} = \mathcal{X} \times \mathcal{X}$, which is obviously the greatest (with respect to inclusion) equivalence relation on $\mathcal{X}$: $Q \subseteq T_{\mathcal{X}}$.

3.1.2 Equivalence classes

For a fixed element $x \in \mathcal{X}$ consider a collection $K(x)$ of all elements of $\mathcal{X}$ that are equivalent to x:

$$K(x) = \{y \in \mathcal{X} : \; y \sim x\};$$

this set is called equivalence class of x. Obviously $y \in K(x)$ implies $K(y) = K(x)$, that is, equivalence class $K(x)$ may be produced by any of its elements. If $z \notin K(x)$, then $K(z)$ and $K(x)$ do not intersect (if this is not the case, then transitivity is violated). That is why each point $x \in \mathcal{X}$ belongs to one and only one equivalence class. This leads to

Theorem 3.1 *Let $\sim$ be equivalence relation on a set $\mathcal{X}$. Then there is the partition of $\mathcal{X}$ into union of disjoint sets*[1]

$$\mathcal{X} = \sum_{\lambda \in \Lambda} K_\lambda, \tag{1}$$

where each set K_λ is equivalence class.

A collection of all equivalence classes is called factor set of $\mathcal{X}$ with respect to equivalence relation $\sim$; it is denoted by $\mathcal{X}/\sim = \{K_\lambda, \; \lambda \in \Lambda\}$. We will also use more concise notation $\widetilde{\mathcal{X}} = \mathcal{X}/\sim$. A mapping $K : \mathcal{X} \to \widetilde{\mathcal{X}}$ that transforms an element $x \in \mathcal{X}$ to its equivalence class $K(x)$ is called canonical.

[1]We use $\sum$ instead of $\bigcup$ symbol for union of disjoint sets

Equality relation $I_{\mathcal{X}}$ possesses single-point equivalence classes $K(x) = \{x\},\ x \in \mathcal{X}$, thus factor set $\widetilde{\mathcal{X}}$ essentially coincides with $\mathcal{X}$; partition of $\mathcal{X}$ into equivalence classes is most detailed in this case. The greatest equivalence relation $T_{\mathcal{X}}$ possesses the unique equivalence class $K(x) = \mathcal{X}$, $x \in \mathcal{X}$, leading to roughest partition. All meaningful equivalence relations Q lie between these extremal cases: $I_{\mathcal{X}} \subseteq Q \subseteq T_{\mathcal{X}}$.

3.1.3 Examples of equivalence relation

Let $\mathcal{X}$ be a set of integers. Two numbers $m, n \in \mathcal{X}$ are called equivalent if the difference $m - n$ is divided by 2. In this case $\mathcal{X}$ is partitioned into two equivalence classes: sets of even and odd numbers. A more general example: for any integer $k \geq 2$ let us call $m, n \in \mathcal{X}$ equivalent if $m - n$ is divided by k. This equivalence relation leads to a partition of $\mathcal{X}$ into k equivalence classes K_i, $i = 0, 1, \dots, k-1$ such that each class K_i contains numbers of the form $jk + i$, $j \in \mathcal{X}$, that is, numbers giving remainder i when divided by k.

Let X be finite set and $\mathcal{X} = 2^X$ be the set of all subsets of X. In this example we call sets $x, y \in \mathcal{X}$ equivalent: $x \sim y$ if and only if they have the same number of elements: $|x| = |y|$. Each equivalence class $C_X^k,\ k = 0, 1, \dots, |X|$ consists of sets of fixed power k:

$$C_X^k = \{x \subseteq X |\ |x| = k\}.$$

3.2 Order

Definition 3.2 *Reflexive, transitive and antisymmetric relation on $\mathcal{X}$ is called (partial)* order *and is denoted by $\leq$. If this relation is also complete, the order is called* linear.

Together with $\leq$ relation we will consider derivative relation $<$, which is defined by

$$x < y \iff x \leq y,\ y \not\leq x.$$

Standard order on a set of reals $\mathbf{R}$ is an example of linear order. Less trivial example is provided by componentwise (partial) order in $\mathbf{R}^n$: for $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in$

$\mathbf{R}^n$

$$x \le y \iff x_i \le y_i,\ i = 1, \dots, n.$$

The set of all subsets $\mathcal{X} = 2^X$ of a fixed set X may be partially ordered by inclusion: for $x, y \subseteq X$

$$x \le y \iff x \subseteq y.$$

In risk theory we would be most interested in orders on a set of probability distributions. Let $\mathcal{F}$ be the set of all distribution functions on $\mathbf{R}$. The following partial order may be defined on $\mathcal{F}$.

Definition 3.3 Stochastic dominance. *$F \in \mathcal{F}$ precedes $G \in \mathcal{F}$ in the sense of stochastic dominance (or G stochastically dominates F): $F \le_1 G$, if*

$$F(x) \ge G(x),\ x \in \mathbf{R}.$$

Consider several examples. Let F_X be distribution function of a random variable X. If $a > 0$ then distribution of $X + a$ stochastically dominates that of X:

$$F_X \le_1 F_{X+a},\ a > 0.$$

For $a \in \mathbf{R}$ denote W_a a degenerate random variable $\mathbf{P}\{W_a = a\} = 1$. Stochastic dominance on the set of all degenerate distribution functions

$$\{F_{W_a},\ a \in \mathbf{R}\}$$

corresponds to natural linear order in $\mathbf{R}$:

$$F_{W_a} \le_1 F_{W_b} \iff a \le b.$$

For reals $a < b$ and $p \in [0, 1]$ denote $B(a, b, p)$ Bernoulli random variable

$$\mathbf{P}\{B(a, b, p) = a\} = 1 - p,\ \mathbf{P}\{B(a, b, p) = b\} = p,$$

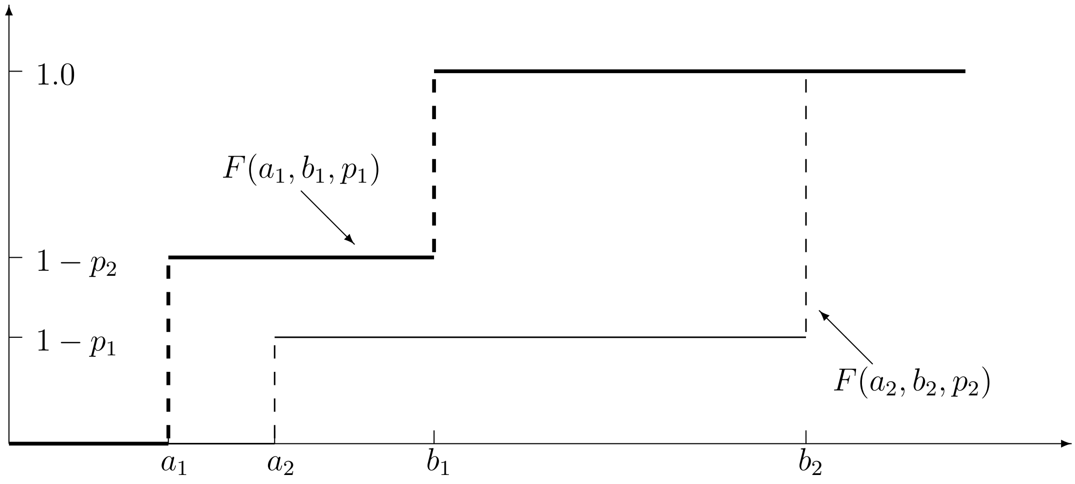

Figure 1: Stochastic dominance of Bernoulli distributions

and $F(a, b, p)$ – its distribution function. Then (figure 1)

$$F(a_1, b_1, p_1) \leq_1 F(a_2, b_2, p_2), \;\; a_1 \leq a_2, \; b_1 \leq b_2, \; p_1 \leq p_2.$$

It is worth noting that if $b_1 \leq a_2$, then $F(a_1, b_1, p_1) \leq_1 F(a_2, b_2, p_2)$ for all p_1, p_2.

3.3 Preference

Definition 3.4 *Complete transitive relation $\preceq$ on a set $\mathcal{X}$ is called preference relation.*

Next figure presents a preference relation on the set $\mathcal{X} = \{x, y, z, t\}$.

	x	y	z	t
x	○	·	·	·
y	○	○	○	·
z	○	○	○	·
t	○	○	○	○

(2)

We would be interested in connection between order and preference relations on the same set $\mathcal{X}$.

Definition 3.5 *Let an order $\leq$ and a preference $\preceq$ relations be defined on a set $\mathcal{X}$. Preference $\preceq$ is called* concordant *with $\leq$ if*

$$x \leq y \implies x \preceq y.$$

For example, preference in (2) is concordant with linear order $x \leq y \leq z \leq t$ and each of its suborders.

Next lemma shows that each preference relation on a set $\mathcal{X}$ generates equivalence relation on the same set.

Lemma 3.2 *Let $\preceq$ be a preference relation on $\mathcal{X}$. Then the relation $\sim$, defined by*

$$x \sim y \iff x \preceq y,\, y \preceq x, \tag{3}$$

is equivalence relation on $\mathcal{X}$, id est it is symmetric, transitive and reflexive.

Proof of this lemma is left to reader (exercise 5.3). $\diamond$

In example (2) we have $y \sim z$ (and of course, each element x, y, z, t is equivalent to itself).

It is natural to consider also a derivative relation of strict preference $\prec$ on $\mathcal{X}$:

$$x \prec y \iff x \preceq y,\; y \not\preceq x.$$

In example (2) we have $x \prec y \sim z \prec t$.

Let $\widetilde{\mathcal{X}}$ be a factor-set $\mathcal{X}$ with respect to equivalence (3), and K be a canonical mapping of $\mathcal{X}$ onto $\widetilde{\mathcal{X}}$. In example (2) we have

$$K(x) = \{x\},\; K(y) = K(z) = \{y, z\},\; K(t) = \{t\}.$$

It is easy to see that preference relation $\preceq$ generates a linear order on $\widetilde{\mathcal{X}}$:

$$K(x) \leq K(y) \iff x \preceq y,$$

and

$$K(x) < K(y) \iff x \prec y.$$

In example (2)

$$K(x) < K(y) = K(z) < K(t).$$

4 Monotone functionals

Definition 4.1 *Let* $\leq$ *be an order on a set* $\mathcal{X}$*. A functional* $\mu : \mathcal{X} \to \mathbf{R}$ *is called nondecreasing if*

$$x \leq y \implies \mu(x) \leq \mu(y).$$

Concepts of nonincreasing, increasing and decreasing functionals are defined similarly; all of them are called monotone (with respect to order $\leq$).

Definition 4.2 *Let* $\preceq$ *be a preference relation on a set* $\mathcal{X}$*. Functional* $\mu : \mathcal{X} \to \mathbf{R}$ *is called nondecreasing if*

$$x \preceq y \implies \mu(x) \leq \mu(y).$$

Concepts of nonincreasing, increasing and decreasing functionals are defined similarly; all of them are also called monotone (with respect to preference $\preceq$).

A connection between concepts of monotonicity is stated in the following

Proposition 4.1 *Let* $\leq$ *be an order on a set* $\mathcal{X}$*, and* $\preceq$ *be a preference relation on the same set, concordant with* $\leq$*. Then each functional* $\mu : \mathcal{X} \to \mathbf{R}$ *that is monotone with respect to* $\preceq$ *in some sence, is monotone with respect to* $\leq$ *in the same sence.*

Proof is left to reader (exercise 5.4).

5 Exercises

Exercise 5.1 *Prove that each complete relation is reflexive.*

Exercise 5.2 *Make up examples of relations that:*

- *are complete but neither symmetric nor transitive nor antisymmetric;*
- *are symmetric but neither transitive nor reflexive nor antisymmetric;*

- *illustrate lack of all other interconnections mentioned in section 2.3.*

Exercise 5.3 *Prove lemma 3.2.*

Exercise 5.4 *Prove proposition 4.1, giving proper treatment of words "in the same sence" in advance.*