Krasnoyarsk State University

Chair of Applied Mathematics

Risk Theory

Topic: Basic Concepts of Risk Theory

Lecture for math department students

A.Novosyolov

Institute of Computational Modeling SB RAS,

Academgorodok, Krasnoyarsk, Russia, 660036

e-mail: arcady@novosyolov.com, phone +7 960 765 1562

Last changed January 4, 2002

©A.Novosyolov, 2002

Krasnoyarsk 2002

Contents

1 Introduction **2**

2 Basic concepts **3**

2.1 Decision-making problem 3

2.2 Risk 4

2.3 Preference relation 4

2.4 Risk measure 5

2.4.1 Examples of risk measures 6

3 Typical applications of risk theory **7**

3.1 Portfolio analysis 7

3.2 Insurance 8

4 Exercises **8**

Abstract

Basic concepts of risk theory as decision making under probabilistic uncertainty are introduced. A decision making problem is stated, some research directions are identified and application areas are pointed to.

1 Introduction

It is perhaps very difficult to find a single field of human activities where it is possible to make sure and certain prediction of future events. Uncertainty may concern the very occurence of an event, time of its occurence, some quantitative characteristics of an event, and so on. In spite of the uncertainty we are to make decisions every day, taking the risk of mistakes due to uncertainty, because results of our decisions are influenced not only by decisions, but also by many external factors, which we would call altogether "state of environment".

Risk theory is a theory of decision-making under probabilistic uncertainty. In this lecture we will introduce basic concepts of the theory, and consider some typical problems and application areas.

2 Basic concepts

In this section we will introduce basic concepts of risk theory [1].

2.1 Decision-making problem

Let $\mathcal{S}$ be the set of all possible states of environment, $\mathcal{D}$ be the set of all available decisions, $\mathcal{R}$ be the set of achievable results. If a decision $d \in \mathcal{D}$ has been made, and environment occured in the state $s \in \mathcal{S}$, then the decision leads to a result $r \in \mathcal{R}$, which is calculated by $r = G(s, d)$, where $G: \mathcal{S} \times \mathcal{D} \to \mathcal{R}$ is a mapping from Cartesian product $\mathcal{S} \times \mathcal{D}$ to $\mathcal{R}$.

It is usually easy to define a natural order or preference relation $\preceq$ on result set $\mathcal{R}$, so that for each pair of results $r_1, r_2 \in \mathcal{R}$ one can tell which of results is better: $r_1 \preceq r_2$ or $r_2 \preceq r_1$, or they are equally preferable: $r_1 \preceq r_2$ and $r_2 \preceq r_1$. If each decision $d \in \mathcal{D}$ would lead to a certain result $r = \widetilde{G}(d) \in \mathcal{R}$, then we could compare decisions d_1, d_2 by comparing their results $\widetilde{G}(d_1), \widetilde{G}(d_2)$. Since environment influences the result as well, the problem becomes more difficult.

Let $\mathcal{S}$ be endowed with a σ-algebra $\mathcal{A}$ and a probability measure $\mathbf{P}_{\mathcal{S}}$ so that triple $(\mathcal{S}, \mathcal{A}, \mathbf{P}_{\mathcal{S}})$ forms a probability space that serves a model of uncertainty of environment states. Let $\mathcal{R}$ be also endowed with σ-algebra $\mathcal{B}$, so that a pair $(\mathcal{R}, \mathcal{B})$ form a measurable space. We will assume that given fixed $d \in \mathcal{D}$, the mapping $G_d: \mathcal{S} \to \mathcal{R}$, defined by $G_d(s) = G(s, d)$, $s \in \mathcal{S}$, is measurable with respect to pair of σ-algebras $\mathcal{A}, \mathcal{B}$ (this can always be achieved by choosing rich enough σ-algebra $\mathcal{A}$). Now G_d defines a random element on $(\mathcal{S}, \mathcal{A}, \mathbf{P}_{\mathcal{S}})$, which induces a probability distribution $\mathbf{P}_d$ on $(\mathcal{R}, \mathcal{B})$:

$$\mathbf{P}_d(B) = \mathbf{P}_{\mathcal{S}}(G_d^{-1}(B)), \quad B \in \mathcal{B}.$$

In what follows it does not matter how distributions $\mathbf{P}_d$ on $(\mathcal{R}, \mathcal{B})$ appear, so we would just assume that there is a family of distributions

$$\mathcal{P}_{\mathcal{D}} = \{\mathbf{P}_d,\ d \in \mathcal{D}\},$$

and would not consider $\mathcal{S}$ explicitly.

Now we can see that a decision d leads to a distribution on $\mathcal{R}$ rather than to a certain result $r \in \mathcal{R}$. That is why it is not sufficient to have an order (preference) on $\mathcal{R}$; the set of distributions should be ordered some way.

2.2 Risk

Denote $\mathcal{P}$ the set of all probability distributions on $(\mathcal{R}, \mathcal{B})$.

Definition 2.1 *Risk is any distribution* $\mathbf{P} \in \mathcal{P}$.

If result set is contained in reals: $\mathcal{R} \subseteq \mathbf{R}$, then risks are distributions of random variables, which may be described by real distribution functions [2]. Other examples are distributions of random vectors in $\mathbf{R}^n$, random processes [3], and random sets [4].

Clearly the set of distributions $\mathcal{P}_{\mathcal{D}}$, produced by decisions, lies within $\mathcal{P}$. So to compare decisions it is sufficient to be able to compare distributions in $\mathcal{P}$ (risks). Since two quite different distributions may occur equally "good" from the decision-making point of view, it is natural to use preference relation on $\mathcal{P}$ rather than order relation.

2.3 Preference relation

Recall preference relation concept [5].

Definition 2.2 *A relation* $\preceq$ *on a set* $\mathcal{R}$ *is called preference relation, if it has the following two properties:*

a) completeness: for each pair $r_1, r_2 \in \mathcal{R}$ *at least one of* $r_1 \preceq r_2$, $r_2 \preceq r_1$ *is true;*

b) transitivity: $r_1 \preceq r_2$ *and* $r_2 \preceq r_3$ *imply* $r_1 \preceq r_3$.

Elements $r_1, r_2 \in \mathcal{R}$, for which both $r_1 \preceq r_2$, $r_2 \preceq r_1$, are called equivalent $r_1 \sim r_2$. It is straightforward that the latter relation is symmetric, reflexive and transitive, so it is actually equivalence relation [5].

Consider also strict preference $\prec$, defined by

$$r_1 \prec r_2 \Longleftrightarrow r_1 \preceq r_2,\ r_2 \not\preceq r_1.$$

We will also use self evident notation $\succeq$ and $\succ$.

2.4 Risk measure

A usual way of setting preference relation on risks set $\mathcal{P}$ is defining a risk measure, that is, a functional on $\mathcal{P}$.

Definition 2.3 *Risk measure is a functional*

$$\mu :\ \mathcal{P} \to \mathbf{R}. \tag{1}$$

Given a functional (1), an induced preference relation $\preceq_\mu$ may be written as

$$P_1 \preceq_\mu P_2 \Longleftrightarrow \mu(P_1) \leq \mu(P_2) \tag{2}$$

or

$$P_1 \preceq_\mu P_2 \Longleftrightarrow \mu(P_1) \geq \mu(P_2) \tag{3}$$

(see exercise 4.1).

Of course, using arbitrary functional μ in decision-making would most probably lead to poor results. A "good" risk measure should reflect preference relation of decision-maker (individual or institutional). In particular it should be monotone with respect to natural orderings of risks set $\mathcal{P}$ [5]. This topics will be considered in detail in further lectures. Here we will present some examples of risk measures for real distributions.

2.4.1 Examples of risk measures

Let $\mathcal{R} \subseteq \mathbf{R}$, then $\mathcal{P}$ is a set of distribution functions on $\mathbf{R}$. An ancient example is expectation

$$\varepsilon(F) = \int_{-\infty}^{\infty} x \, dF(x), \quad F \in \mathcal{P}. \tag{4}$$

This risk measure is often used until now, when decision is made using only mean values paying no attention to uncertainty. If environment uncertainty is essential, such a decision-making may lead to great, sometimes catastrofic, mistakes.

Another example is variance of a distribution

$$\delta(F) = \int_{-\infty}^{\infty} (x - \varepsilon(F))^2 \, dF(x), \quad F \in \mathcal{P}. \tag{5}$$

This risk measure allows taking uncertainty into account; it is the base for Markowitz model [6], and Sharpe's CAPM (Capital Asset Pricing Model).

Mixture of ε and δ is also a risk measure:

$$\gamma(F) = \delta(F) - \beta\varepsilon(F), \quad F \in \mathcal{P}; \tag{6}$$

here $\beta > 0$ is a weighting parameter.

Modern applications make intensive use of expected utility measure

$$\rho(F) = \int_{-\infty}^{\infty} U(x) \, dF(x), \quad F \in \mathcal{P}, \tag{7}$$

where U is some real function.

Recently a distorted probability measure was introduced [7], [8]:

$$\pi(F) = \int_{-\infty}^{0} [g(1 - F(x)) - 1] \, dx + \int_{0}^{\infty} g(1 - F(x)) \, dx, \quad F \in \mathcal{P}, \tag{8}$$

where $g : [0, 1] \to [0, 1]$ is a monotone function with $g(0) = 0$, $g(1) = 1$.

In application the Value-at-risk (VaR) measure ν is actively used, it is an α quantile of a distribution:

$$\nu(F) = F^{-1}(\alpha). \tag{9}$$

This risk measure is a special case of (8) with

$$g(v) = \begin{cases} 0, & v < 1-\alpha, \\ 1, & v \geq 1-\alpha. \end{cases} \tag{10}$$

Remark that risk measures ε and δ are fixed, risk measures γ and ν possess some flexibility due to parameters α β, and risk measures ρ and π are highly flexible; the latter property allows choosing appropriate risk measure for specific decision-makers.

3 Typical applications of risk theory

Now consider examples of risk theory applications.

3.1 Portfolio analysis

Let $X = (X_1, \ldots, X_n)$ be an n-dimensional random vector, the component X_i being the return on investing a unit capital into i-th investment tool, $i = 1, \ldots, n$. Portfolio selection problem consists in choosing the best (in the sence of a given preference relation) mode of partitioning the unit capital among investment tools. Assuming linear relation between capital invested and investment return for each tool, portfolio return may be written as

$$Y = y_1 X_1 + \ldots + y_n X_n,$$

where $y = (y_1, \ldots, y_n)$ is a vector of capital portions (weights), invested in corresponding tools.

The situation just described may be formalized as a decision-making problem as follows. The set of environment states $\mathcal{S}$ is a subset in $\mathbf{R}^n$; this is the range of values of the random vector X. Probability space on $\mathcal{S}$ is formed by Borel σ-algebra and probability distribution of X. The set of decisions $\mathcal{D}$ is

$$\mathcal{D} = \{y = (y_1, \ldots, y_n) \in \mathbf{R}^n |\ y_1 \geq 0, \ldots, y_n \geq 0;\ \sum_{i=1}^{n} y_i = 1\},$$

the standard simplex in $\mathbf{R}^n$. The set of results $\mathcal{R}$ lies in $\mathbf{R}$, each risk is a distribution function F_Y of a random variable Y, portfolio return. Preference relation is induced by a

risk measure, and decision-making is choosing a point in $\mathcal{D}$ that provides extremal value for the risk measure. Since distribution of Y depends on $y \in \mathcal{D}$, the value of risk measure also depends on y: $\mu(F_Y) = f(y)$, thus portfolio optimization problem gets the form

$$f(y) \longrightarrow \max_{y \in \mathcal{D}} \ (\min_{y \in \mathcal{D}}). \tag{11}$$

3.2 Insurance

A standard problem of insurance is premium calculation given a distribution F_Y of future insurance loss Y. Solution of the problem also reduces to choice of a risk measure μ so that premium P calculation is made by

$$P = \mu(F_Y). \tag{12}$$

Choosing a proper risk measure μ is being actively discussed [7], [9]; we will consider this problem in detail in subsequent lectures.

4 Exercises

Exercise 4.1 *Prove that (2), (3) actually define preference relation in the sence of definition 2.2.*

Exercise 4.2 *For $a \in \mathbf{R}$ denote W_a a random variable with degenerate distribution at a: $\mathbf{P}\{W_a = a\} = 1$. For $a, b \in \mathbf{R}$, $a < b$, $p \in [0,1]$ denote $B(a,b,p)$ a Bernoulli random variable*

$$\mathbf{P}\{B(a,b,p) = a\} = 1 - p, \ \ \mathbf{P}\{B(a,b,p) = b\} = p.$$

Calculate values of risk measures ε, δ, γ, ρ, π and ν for distributions of these variables. Give the precise definition of a quantile (9) so that it is appropriate for discontinuous distribution functions. Choose appropriate extremum type (maximum or minimum) to be used in (11) with each of the risk measures.

Exercise 4.3 *Prove that (9) is a special case of (8) with g of the form (10).*

Exercise 4.4 *Show that if a risk measure μ is monotone with respect to an order $\leq$, then induced preference relation $\preceq_\mu$ is concordant [5] with the order $\leq$.*

References

[1] A.A. Novosyolov (2001) *Mathematical modeling of financial risks. Measuring theory.* Novosibirsk: Nauka, 102 p. (in Russian)

[2] W. Feller (1971) *Introduction to probability theory and its applications.*- N.Y.: Wiley.

[3] J.L. Doob (1953) *Stochastic processes.*- N.Y.: Wiley.

[4] O.Y. Vorobyov (2002) *Random finite abstract sets.* Novosibirsk: Nauka, to appear.

[5] A.A. Novosyolov (2002) Relations. *Lecture for students of KSU (Mathematical department).*
http://www.geocities.com/novosyolov/lectures.htm

[6] A.A. Novosyolov (2002) Investment portfolio selection, *Lecture for students of KSU (Mathematical department).*
http://www.geocities.com/novosyolov/lectures.htm

[7] Wang, S. (1996) Premium calculation by transforming the layer premium density. *ASTIN Bulletin*[1], **26**, pp. 71-92.

[8] Young V.R. (1999) Discussion of Christofides' Conjecture Regarding Wang's Premium Principle. *ASTIN Bulletin*, **29**, 2, 191–195.

[9] Buhlmann Hans (1996) *Mathematical Methods in Risk Theory.* Springer, Berlin.

[1]Full texts of papers from ASTIN Bulletin are available online as PDF files at http://www.casact.org/library/astin/