Krasnoyarsk State Technological University

Risk Theory

Topic: Simple insurance portfolios

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1 Introduction

Calculating insurance premium is one of basic problems faced by an insurance company. The premium size has significant impact on company's ability to meet its liabilities and stability of company functioning. In the present lecture premium calculation is being considered within a very simple portfolio framework; premium size is selected as a solution to the following equation

$$\mathbf{P}\{C \le Q\} = \alpha, \tag{1}$$

where C is a random variable representing total claim size of a portfolio, Q is total portfolio premium, and α is a safety bound, a cover probability that is set by an insurance company, $\alpha \in [0, 1]$. In most cases $0.8 \le \alpha \le 0.95$. Premium is represented as a percentage T of policy liability.

Below we will consider three types of portfolios called "simplest", "simple" and "real" here, each subsequent portfolio type being a generalization of the previous one. Each portfolio consists of N similar policies; each policy produces claim with probability p. Differences among the types are due to claim amount calculation and are specified below.

2 Premium calculation

General scheme for premium calculation is as follows. Let Q be the total premium size of a portfolio, and let C be its total claim amount. Denote F_C the distribution function of the random variable C, and let F be the distribution function of the normalized variable

$$C^0 = \frac{C - \mathbf{E}C}{\sqrt{\mathbf{D}C}},$$

where $\mathbf{E}C$ and $\mathbf{D}C$ represent expectation and variance of C. Then (1) is equivalent to

$$\alpha = \mathbf{P}\left\{\frac{C - \mathbf{E}C}{\sqrt{\mathbf{D}C}} \le \frac{Q - \mathbf{E}C}{\sqrt{\mathbf{D}C}}\right\} = \mathbf{P}\left\{C^0 \le \frac{Q - \mathbf{E}C}{\sqrt{\mathbf{D}C}}\right\} = F\left(\frac{Q - \mathbf{E}C}{\sqrt{\mathbf{D}C}}\right),$$

which leads to

$$Q = \mathbf{E}C + \sqrt{\mathbf{D}C}F^{-1}(\alpha), \tag{2}$$

where F^{-1} denotes an inverse function of F.

If distribution of C is not known and portfolio volume N is large, the central limit theorem states that distribution of C is approximately normal, so (2) may be substituted by

$$Q = \mathbf{E}C + \sqrt{\mathbf{D}C}\Phi^{-1}(\alpha), \tag{3}$$

where Φ is a standard normal distribution function.

2.1 Simplest portfolio

Consider a portfolio of N policies, each rasing a claim of size S with probability p. Thus, premium size for each policy equals TS, and the total premium size for a portfolio is calculated as

$$Q = TSN. \tag{4}$$

Individual i-th claim size is described by a random variable C_i with the following distribution:

Value	0	S
Probability	$1 - p$	p

,

and total portfolio claim size C is obtained by summing up individual claims:

$$C = C_1 + ... + C_N.$$

Assuming individual claims mutually independent, we get the binomial distribution for C:

Value	0	S	$2S$	...	NS
Probability	p_0	p_1	p_2	...	p_n

, (5)

where

$$p_k = C_N^k p^k (1-p)^{N-k}, \quad k = 0, 1, ..., N.$$

It is well known that binomial distribution (5) possesses expectation and variance of the form

$$\mathbf{E}C = NSp, \quad \mathbf{D}C = NS^2 p(1-p). \tag{6}$$

Inversion of a distribution function of C for large N may be troublesome, so using normal approximation we get from (3), (4) and (6):

$$TSN = pSN + \sqrt{NS^2 p(1-p)}\Phi^{-1}(\alpha),$$

and premium takes the form

$$T = p + \sqrt{\frac{p(1-p)}{N}}\Phi^{-1}(\alpha) \tag{7}$$

$$= p\left(1 + \sqrt{\frac{1-p}{Np}}\Phi^{-1}(\alpha)\right). \tag{8}$$

Expression (7) shows that premium consists of two parts

$$T_0 = p, \quad T_r = \sqrt{\frac{p(1-p)}{N}}\Phi^{-1}(\alpha),$$

that may be called basic premium and risk loading. Second term in parentheses in (8) is equal to

$$T_{rr} = T_r / T_0,$$

and represents relative risk loading.

Dependence of premium (7), (8) on parameters is shown in section 3 below.

2.2 Simple portfolio

Now consider a simple portfolio. It is almost the same as the simplest one, but policies liabilities may be different. Thus, simple portfolio is defined by parameters N, p, α with the same sense as before, and a sequrnce of liabilities $S_1, ..., S_N$. For convenience let

$$\overline{S} = \frac{1}{N}\sum_{i=1}^{N} S_i, \quad \overline{\overline{S}} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} S_i^2}. \tag{9}$$

Denoting premium percentage T as before, we get i-th premium size as TS_i, and total portfolio premium becomes equal to

$$Q = T\sum_{i=1}^{N} S_i = TN\overline{S}. \tag{10}$$

Claim size C_i of a policy i is a random variable taking value S_i with probability p, and value 0 with probability $1-p$, so its mean and variance are equal to $\mathbf{E}C_i = pS_i$ and $\mathbf{D}C_i = p(1-p)S_i^2$. Distribution of a portfolio claim size $C = C_1 + ... + C_N$ turns out much more complicated than before, but in case of large portfolio volume N the central limit theorem still works. Expectation and variance of C are easily calculated:

$$\mathbf{E}C = p\sum_{i=1}^{N} S_i = pN\overline{S}, \quad \mathbf{D}C = p(1-p)\sum_{i=1}^{N} S_i^2 = p(1-p)N\overline{\overline{S}}^2,$$

so using (2), (10), one obtains the following equation for T:

$$TN\overline{S} = pN\overline{S} + \overline{\overline{S}}\sqrt{p(1-p)N}\Phi^{-1}(\alpha),$$

thus

$$T = p + \frac{\overline{\overline{S}}}{\overline{S}}\sqrt{\frac{p(1-p)}{N}}\Phi^{-1}(\alpha) \tag{11}$$

$$= p\left(1 + \frac{\overline{\overline{S}}}{\overline{S}}\sqrt{\frac{1-p}{pN}}\Phi^{-1}(\alpha)\right). \tag{12}$$

Basic premium and risk loadings take the form

$$T_0 = p, \;\; T_r = \frac{\overline{\overline{S}}}{\overline{S}}\sqrt{\frac{p(1-p)}{N}}\Phi^{-1}(\alpha), \;\; T_{rr} = \frac{\overline{\overline{S}}}{\overline{S}}\sqrt{\frac{1-p}{pN}}\Phi^{-1}(\alpha).$$

Comparing (11), (12) with corresponding premium expressions (7), (8) for simplest portfolio, it is easy to see that they are different only by a coefficient $\beta(S) = \overline{\overline{S}}/\overline{S}$, where $S = (S_1, .., S_N)$. Let us establish the range of values of the coefficient when the set of liabilities S varies. Denote $\mathbf{R}_+^N = \{S \in \mathbf{R}^N : S_1 \geq 0, ..., S_N \geq 0\}$ the nonnegative orthant of $\mathbf{R}^N$, the set of all possible values of S.

Proposition 2.1

$$1 \leq \beta(S) \leq \sqrt{N},\ S \in \mathbf{R}_+^N. \tag{13}$$

Proof. Note that

$$\beta^2(S) = N \frac{S_1^2 + ... + S_N^2}{(S_1 + ... + S_N)^2}, \tag{14}$$

so $\beta(\gamma S) = \beta(S)$ for any $\gamma > 0$, thus it is sufficient to find upper and lower bounds of $\beta(S)$ for S belonging to the standard simplex of $\mathbf{R}^N$:

$$\mathbf{S} = \{S \in \mathbf{R}_+^N : S_1 + ... + S_N = 1\}.$$

Since denominator of (14) is constant on $\mathbf{S}$, minimizing $\beta(S)$ on $\mathbf{S}$ is equivalent to the problem

$$f(S) = S_1^2 + ... + S_N^2 \to \min_{S \in \mathbf{S}}, \tag{15}$$

$$g(S) = S_1 + ... + S_N - 1 = 0. \tag{16}$$

To apply Lagrange multipliers method, write Lagrange function

$$L(S, \lambda) = f(S) + \lambda g(S)$$

and minimum necessary conditions

$$\frac{\partial L}{\partial S_i} = 2S_i + \lambda = 0, \quad i = 1, ..., N, \tag{17}$$

$$\frac{\partial L}{\partial \lambda} = S_1 + ... + S_N - 1 = 0. \tag{18}$$

Now (17) implies $S_i = -\lambda/2,\ i = 1, ..., N$. Substituting this into (18), we get $\lambda = -2/N$, so

$$S_i^* = \frac{1}{N},\ i = 1, ..., N.$$

Since $S_i^* > 0$, this is the point of minimum of $\beta(S)$ on $\mathbf{S}$. Thus minimal value of $\beta(S)$ is attained in the geometric center of the standard simplex $S^* = (1/N, ..., 1/N)$ and is equal to

$$\beta(S^*) = 1.$$

Symmetry implies that maximal value of $\beta(S)$ is reached in vertices of $\mathbf{S}$ and is equal to

$$\beta(1, 0, ..., 0) = \sqrt{N}.$$

Thus, proof is complete. $\diamond$

Proposition 2.1 means that the coefficient β (and premium size) is minimal when liabilities of policies are equal, and may be significantly greater when there is a dispersion of liabilities in a portfolio. Maximal premium corresponds to the case of single sharply large liability.

2.3 Real portfolio

This portfolio is similar to the simple one; the only difference is claim size, that can be any number in $[0, S_i]$ for i-th policy. To be precise:

$$C_i = \xi_i r S_i,$$

where ξ_i, $i = 1, ..., N$ are Bernoulli random variables, taking values 1 with probability p and 0 with probability $1 - p$ (they are indicators of claim occurrence); r is a random variable with distribution restricted to $[0, 1]$, describing relative claim size. We will assume random variables $r, \xi_1, ..., \xi_N$ to be mutually independent.

Now let expectation and variance of r be known:

$$\mathbf{E}r = m, \quad \mathbf{D}r = \tau^2. \tag{19}$$

Then parameters of an individual claim distribution are

$$\mathbf{E}C_i = pmS_i, \quad \mathbf{D}C_i = S_i^2 pm^2 \left(1 - p + \frac{\tau^2}{m^2}\right) \tag{20}$$

while those of total portfolio claims $C = C_1 + ... + C_N$ are

$$\mathbf{E}C = pmN\overline{S}, \quad \mathbf{D}C = pm^2 \left(1 - p + \frac{\tau^2}{m^2}\right) N\overline{\overline{S}}^2. \tag{21}$$

Portfolio premium becomes $Q = TN\overline{S}$ as before, so (21), (2) lead to equation

$$TN\overline{S} = pmN\overline{S} + m\overline{\overline{S}}\sqrt{p(1 - p + \tau^2/m^2)N}\Phi^{-1}(\alpha),$$

which imply

$$T = pm + m\frac{\overline{\overline{S}}}{\overline{S}}\sqrt{\frac{p(1 - p + \tau^2/m^2)}{N}}\Phi^{-1}(\alpha) \tag{22}$$

$$= pm\left(1 + \frac{\overline{\overline{S}}}{\overline{S}}\sqrt{\frac{1 - p + \tau^2/m^2}{pN}}\Phi^{-1}(\alpha)\right) \tag{23}$$

Basic premium and risk loadings are

$$T_0 = pm, \;\; T_r = m\frac{\overline{\overline{S}}}{\overline{S}}\sqrt{\frac{p(1 - p + \tau^2/m^2)}{N}}\Phi^{-1}(\alpha), \;\; T_{rr} = \frac{\overline{\overline{S}}}{\overline{S}}\sqrt{\frac{1 - p + \tau^2/m^2}{pN}}\Phi^{-1}(\alpha).$$

Comparing (22), (23) with corresponding expressions for a simple portfolio (11), (12), one can easily see that basic premium is now equal to pm, the average claim size, while risk loading contains additional term τ^2/m^2, representing uncertainty arising from claim size dispersion.

3 Illustrations

Note that some standard techniques suggest premium calculation using formula (22) with coefficient $\beta = \overline{\overline{S}}/\overline{S}$ substituted by a constant 1.2. It is interesting to establish how large might be deviations of β from that value under realistic assumptions. To answer the question suppose that policies liabilities are drawn randomly from a distribution with CDF G. In this case β is a random variable, and its expectation $\mu = \mathbf{E}\beta$ and standard deviation $\sigma = \sqrt{\mathbf{D}\beta}$ may be easily estimated via Monte Carlo method. Next table contains these results for several distributions G, $N = 20$ and number of Monte Carlo trials equal to 10,000. One can see that heavy tailed distributions G provide significantly larger coefficients β than 1.2.

Distribution G	μ	σ
Uniform on $[0, A]$	1.154	0.048
Binomial, $n = 10, p = 0.3$	1.107	0.035
Binomial, $n = 20, p = 0.05$	1.395	0.135
Lognormal, $\mu = 1, \nu = 0.3$	1.043	0.015
Lognormal, $\mu = 1, \nu = 1$	1.454	0.229
Lognormal, $\mu = 1, \nu = 2$	2.288	0.598