Krasnoyarsk state university

Chair of applied mathematics

Risk theory

Topic: Uniform distribution on a standard simplex in $\mathbf{R}^n$

Lecture for students of math department of KSU

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Abstract

A method of simulating a uniform distribution on a standard simplex in $\mathbf{R}^n$ is considered in the lecture.

Contents

1 Introduction

Optimization problems, in particular in portfolio selection, are often solved by Monte Carlo method, that requires simulation of uniform distrubution on a standard simplex in $\mathbf{R}^n$:

$$S_n = \{y = (y_1, ..., y_n) \in \mathbf{R}^n |\ y_1, ..., y_n \geq 0,\ y_1 + ... + y_n = 1\}. \tag{1}$$

Consider the following problem as an example. Let $(X_1, ..., X_n)$ be a random vector, which components represent returns of some assets when investing a unit capital. Then allocation of the unit capital among assets with weights $y_i,\ i = 1, \ldots, n$ would lead to a portfolio return $X = y_1 X_1 + ... + y_n X_n$. A standard portfolio selection problem takes the form

$$f(y_1, ..., y_n) = H(X) \to \max_{(y_1,...,y_n)} \quad (\min_{(y_1,...,y_n)}); \tag{2}$$

subject to constraints

$$y_1, ..., y_n \geq 0; \quad y_1 + ... + y_n = 1, \tag{3}$$

where H is a risk measure.

Below a method of simulating a uniform distribution on a standard simplex S_n is presented and justified. For future reference denote $e_i = (0, ..., 0, 1, 0, ..., 0),\ i = 1, ..., n$ unit vectors in $\mathbf{R}^n$, $Co(A)$ a convex hull of a set $A \subseteq \mathbf{R}^n$, and

$$J = (1, 1, ..., 1) \in \mathbf{R}^n. \tag{4}$$

2 Simulation of uniform distribution

The method is essentially a transform of uniform distribution on a unit hypercube onto a standard simplex. Below we describe the transform step by step. The schematic of the transform is the following

$$\begin{array}{ccccccccc} C_{n-1} & \xrightarrow{G} & \overline{V}_I & \xrightarrow{F} & W & \xrightarrow{M} & Y & \xrightarrow{P} & S_n \\ U & \to & T & \to & R & \to & Z & \to & Q \end{array}, \tag{5}$$

where the top line presents sets on which uniform distribution is obtained on each step, and corresponding step transforms. Bottom line indicates random variables possessing uniform distribution on corresponding sets.

2.1 Distribution on C_{n-1}

Denote $C_{n-1} = [0,1]^{n-1}$ a unit hypercube in $\mathbf{R}^{n-1}$. Uniform distribution on C_{n-1} is easily obtained as a random vector

$$U = (U_1, ..., U_{n-1}) \tag{6}$$

with independent components, each uniformly distributed on $[0,1]$.

2.2 Distribution on V_I

Let Π be a collection of all permutations $\pi = (i_1, ..., i_{n-1})$ on a set $\{1, 2, ..., n-1\}$, and $I = (1, 2, ..., n-1)$ be the identity. For a permutation $\pi = (i_1, ..., i_{n-1}) \in \Pi$ denote V_π a collection of points $y = (y_1, ..., y_{n-1}) \in C_{n-1}$ possessing the property $y_{i_1} < y_{i_2} < ... < y_{i_{n-1}}$, and $\overline{V}_\pi$ – its closure

$$\overline{V}_\pi = \{y = (y_1, ..., y_{n-1}) | y_{i_1} \leq ... \leq y_{i_{n-1}})\}.$$

In particular, for identity I one obtains

$$V_I = \{y = (y_1, ..., y_{n-1}) |\ y_1 < ... < y_{n-1}\}.$$

By symmetry it is clear that domains $V_\pi,\ \pi \in \Pi$ are congruent, thus possess the same $(n-1)$ – dimensional volume equal to $1/(n-1)!$. Hence the volume of

$$V = \sum_{\pi \in \Pi} V_\pi$$

is equal to 1 and vector U falls into V with probability 1. Remark that $C_{n-1} \setminus V$ consists of points having at least two coinciding coordinates, and $\bigcup_{\pi \in \Pi} \overline{V}_\pi = C_{n-1}$.

Denote $G : C_{n-1} \to \overline{V}_I$ the transform that sorts coordinates of $y \in C_{n-1}$ in ascending order. For a fixed permutation $\pi \in \Pi$ restriction of this transform $G_\pi : V_\pi \to V_I$ on V_π is one-to-one and volume-preserving. Thus

$$T = G(U) \tag{7}$$

possesses the uniform distribution on $\overline{V}_I$, and $T \in V_I$ with probability 1.

2.3 Distribution on W

Denote

$$W = \{y \in \mathbf{R}^{n-1} |\ y_1 \geq 0, ..., y_{n-1} \geq 0,\ \sum_{i=1}^{n-1} y_i \leq 1\} = Co\{0, e_1, ..., e_{n-1}\} \tag{8}$$

and consider a mapping $F: \overline{V}_I \to W$ that maps a point $y = (y_1, ..., y_{n-1}) \in \overline{V}_I$ to a point $y' = (y'_1, ..., y'_{n-1})$ using the rule

$$y'_1 = y_1,$$

$$y'_1 = y_2 - y_1,$$

$$...$$

$$y'_{n-1} = y_{n-1} - y_{n-2}.$$

Clearly $y' = Ay$ with matrix A of the form

$$A = \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ -1 & 1 & 0 & ... & 0 \\ 0 & -1 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & -1 & 1 \end{pmatrix}.$$

F is a linear one-to-one mapping on W with $|A| = 1$, so this is a volume-preserving mapping. Thus

$$R = F(T) \tag{9}$$

possesses a uniform distribution on W.

2.4 Distribution on S_n

For a $y = (y_1, ..., y_{n-1}) \in W$ define the point $y' = E(y) = (y'_1, ..., y'_n) \in \mathbf{R}^n$ by

$$y'_1 = y_1,$$

$$...,$$

$$y'_{n-1} = y_{n-1},$$

$$y'_n = 1 - y_1 - ... - y_{n-1}.$$

Clearly E maps W onto S_n, and is linear: $E(y) = Ay + b$ with

$$A = \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 1 & 0 \\ -1 & -1 & ... & -1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ ... \\ 0 \\ 1 \end{pmatrix}.$$

Linearity of E implies that it keeps $(n-1)$-dimentional volume proportions invariant, thus the distribution on S_n remains uniform. To further clarify the transform from W to S_n let us represent it as a composition of a scaling in $\mathbf{R}^{n-1}$ and a movement in $\mathbf{R}^n$.

2.4.1 Scaling in $\mathbf{R}^{n-1}$

Transform W into $Y \subset \mathbf{R}^{n-1}$ (with Y congruent to S_n) as follows. Note that unit vectors e_i, $i = 1, \dots, n-1$ are equally spaced with distance $\sqrt{2}$, and find the point $A \in \mathbf{R}^{n-1}$ such that its distance from each unit vector is $\sqrt{2}$, and located in the same hemispace defined by the hyperplane

$$\sum_{i=1}^{n-1} y_i = 1, \tag{10}$$

as the origin. Symmetry implies that $A = (a, a, ..., a)$, and $a < 0$. Calculating squared Euclid distance from A to e_1, one obtains $|A - e_1|^2 = 2$, which leads to equation

$$(n-1)a^2 - 2a - 1 = 0.$$

The latter has the only negative root

$$a = \frac{1 - \sqrt{n}}{n-1}.$$

The simplex $Y = Co\{A, e_1, ..., e_{n-1}\}$ represents the required set which is congruent to S_n. Note that the point $A = (a, ..., a)$ is $\beta = \sqrt{n}$ times farther from C, the center of $S_{n-1} = Co\{e_1, ..., e_{n-1}\}$, than the origin (see figure 1).

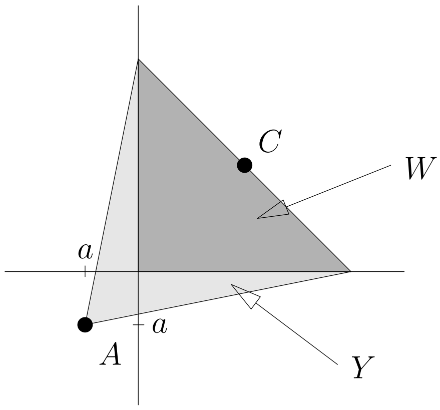

Figure 1: Transform W to Y

Now let us construct M, an affine transform from W to Y. For any point $y = (y_1, ..., y_{n-1}) \in W$ calculate the closest point $\widetilde{y}$ of the simplex S_{n-1}, and set $M(y) = z = \widetilde{y} + \beta(y - \widetilde{y})$. This is clearly an affine transform of W onto Y (see exercise 3.1).

To calculate $\widetilde{y}$ note that it may be represented as $\widetilde{y} = y + \alpha_y J$, where J was defined in (4), and a constant α_y is defined to normilize $\widetilde{y}$ (coordinates of the latter should sum to 1):

$$\alpha_y = \frac{1 - \sum_{i=1}^{n-1} y_i}{n-1}.$$

The transform M is illustrated by figure 1.

Thus

$$M(y) = y - (\beta - 1)\alpha_y J, \tag{11}$$

and a random variable

$$Z = M(R) \tag{12}$$

is uniform on Y.

2.4.2 Distribution on S_n

The last step is the pivoting P of the set Y in $\mathbf{R}^n$, such that S_{n-1} is fixed, and the point A goes to e_n: $P(Y) = S_n$, $P(S_{n-1}) = S_{n-1}$, $P(A) = e_n$. Proof of existence of such a pivoting is left to reader (see exercise 3.2). Since P does not change $(n-1)$-dimentional volume, a random vector

$$Q = P(Z) \tag{13}$$

is uniform on S_n.

A sequence of transforms from C_{n-1} to Y is shown on figure 3 for $n = 3$ and a sample of 20 points. Bullets mark the points that were moved under sorting. Figure 3 shows 500 points simulated in S_3.

2.5 A simple representation of the algorithm

Note that the simulation algorithm may be described in short as follows.

- Draw $n - 1$ uniformly distributed points on $[0, 1]$.
- Set the lengths of n subintervals of $[0, 1]$ thus obtained as coordinates of a vector $y = (y_1, \dots, y_n)$. That's it for a single point $y \in S_n$.
- Repeat the above steps to get as many points on S_n as necessary.

3 Exercises

Exercise 3.1 *Prove that M defined in subsection 2.4.1 is affine: construct a matrix B and a vector c such that $M(y) = By + c$.*

Exercise 3.2 *Write down the pivoting transform mentioned in subsection 2.4.2: $P(y) = By + c$ (construct a matrix B and a vector c).*

References

[1] W.Feller. *Introduction to Probability Theory and its Applications.* **1,2**, New York: Wiley, 1971.

[2] Loeve M. (1960) *Probability theory.* Van Nostrand N.Y.

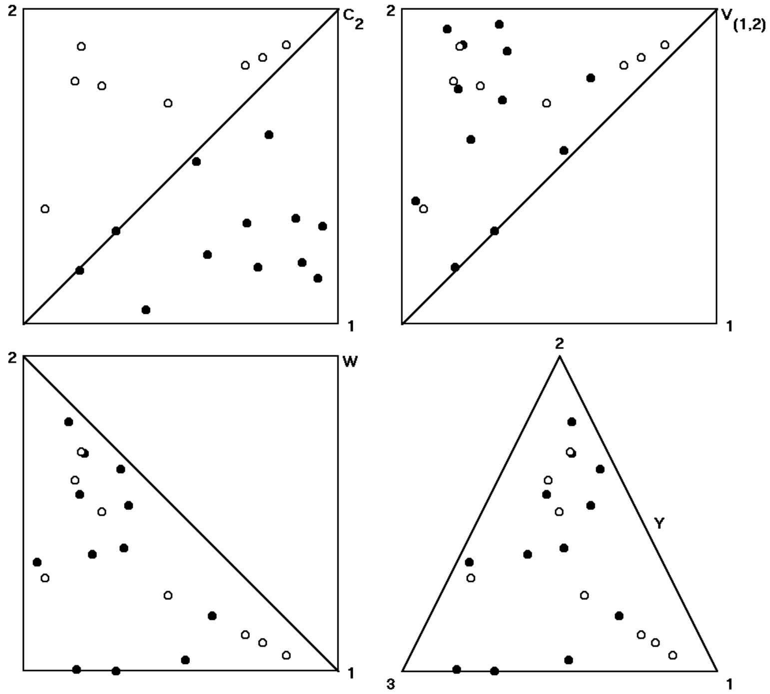

Figure 2: Sequence of transforms

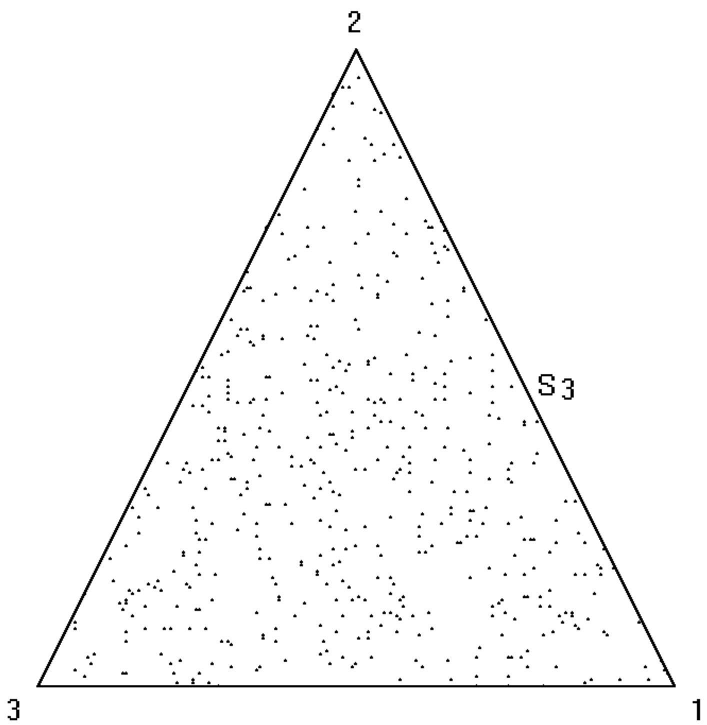

Figure 3: Large sample