Combined functionals as risk measures

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Abstract

Risk measures are widely used in insurance pricing, portfolio selection, and in decision-making in general. Two prevalent classes of risk measures are expected utility (a dollar transform), and distorted probability (a probability transform). Both approaches exhibit properties which are not supported by empirical evidence on decision-making under risk. We propose a combined functional (dollar and probability transform) which may combine advantages of both approaches. The present paper develops representation theorems and axiomatic descriptions, presents applications to decision-making under risk, premium calculation, and portfolio selection; and includes numeric and graphical illustrations.

Key words: risk measure, expected utility, distorted probability, combined functional, premium calculation, portfolio selection, decision-making.

1 Introduction

Measuring risk continues to be both a hot research topic and an important practical issue. An investor might consider the problem: given a project bringing uncertain gain represented by a random variable X, is it worth investing a certain amount a into the project? Or, taken a bit broader: what is the maximum certain amount a that is worth investing into the project? A property owner faces a similar question: given uncertain losses of property X during a specified period of time, is it worth paying an insurance premium a to transfer the risk X? Or what is the maximum premium a that the owner would be willing to pay for insurance? In both cases the certain amount a may be treated as the certainty equivalent of the risk X. This certainty equivalent constitutes a functional on the space of risks (random variables or their distributions), provided that each risk possesses one. The functional (or its strictly monotone transform) is usually called risk measure.

A commonly used risk measure has been $\mathbf{E}X$, the expected value of X. It is actually sometimes used today in life insurance for premium calculation. As long as three hundred years ago Daniel Bernoulli pointed out that using the expected value leads to the so called ”St.Petersburg paradox” [3] (see also [13]); in other words, the expected value principle corresponds to a risk-neutral individual, and works poorly for the more common case of risk averse individuals. Thus the expected value turns out to not be an appropriate risk measure. Bernoulli proposed calculation of the expected utility of X instead of its expected value; this trick may be treated as a dollar transform with the utility function U as a transforming function. Later the expected utility principle received a solid foundation in the book by J. von Neumann and O. Morgenstern [7], which was the first axiomatic construction of a risk measure. Due to this foundation expected utility remains the main tool for decision-making.

In spite of its popularity, expected utility measure has a number of flaws, including linearity with respect to a mixture of distributions. Thus it is not surprising that other risk measures were intensively sought. Recently [14] a distorted probability measure was introduced for insurance pricing. A more general class of coherent risk measures appeared in [2] right after. These happen to be nonlinear in distribution, thus lacking the flaw of expected utility, but unfortunately hold another sort of linearity; they are positive homogeneous, which is also often undesirable.

The present paper is devoted to a combination of expected utility and distorted probability principles. The paper is organized as follows. Section 2 introduces the concepts of risk and risk measure, as well as basic mathematical structures needed in what follows. Section 3 presents methods of defining risk measures via sets of acceptable risks and operators over sets of functionals. Sections 4 and 5 describe expected utility and distorted probability functionals, and analyze their properties and disadvantages. In section 6 a version of the representation theorem for distorted probability measure is stated and proven. Section 7 contains an introduction to the combined functional, including its representation theorem and illustrations, and discusses a resolution of Rabin's paradox [11].

2 Risk measures

In this section we will define a concept of risk measure and present its usage in decision-making problems. First we need to define the concept of risk. In what follows risk is a random variable which represents uncertain gains (losses if negative) of a decision, or its distribution function.

Let $(\Omega, \mathcal{B}, \mathbf{P})$ be a probability space. Consider a set $\mathcal{X}$ of all (almost surely) bounded random variables $X : \Omega \to \mathbf{R}$, which is usually denoted by $\mathcal{X} = L^\infty(\Omega, \mathcal{B}, \mathbf{P})$. Denote $I \in \mathcal{X}$ the identical unity: $I(\omega) = 1$, $\omega \in \Omega$. A (cumulative) distribution function F_X for a risk X is defined as $F_X(x) = \mathbf{P}(\omega \in \Omega : \ X(\omega) \le x)$, $\ x \in \mathbf{R}$, and its inverse is denoted by $F_X^{-1}(v) = \sup\{x : F_X(x) \le v\}$, $v \in [0,1]$. Denote $\mathcal{F}$ the set of all distribution functions with bounded support[1]. Expectation of a random variable X (with respect to the probability measure $\mathbf{P}$) is as usual

$$\mathbf{E}X = \int_\Omega X(\omega)\, d\mathbf{P}(\omega) = \int_{-\infty}^{\infty} t\, dF_X(t) = \int_0^1 F_X^{-1}(v)\, dv = -\int_0^1 F_X^{-1}(v)\, d(1-v). \qquad (1)$$

We will also need expectations with respect to other probability measures Q on the measurable space $(\Omega, \mathcal{B})$:

$$\mathbf{E}_Q X = \int_\Omega X(\omega)\, dQ(\omega).$$

For the sake of simplicity we will often use a finite sample space: $|\Omega| = n$. In this case a probability measure Q on $(\Omega, \mathcal{B})$ is completely defined by an n-tuple $Q = (q_1, \dots, q_n)$ such that $q_1 \ge 0, \dots, q_n \ge 0$ and $q_1 + \dots + q_n = 1$. In other words, the set $\mathcal{F}$ of all probability distributions is represented by a standard simplex S_n in $\mathbf{R}^n$, in particular $\mathbf{P} = (p_1, \dots, p_n)$. A random variable $X \in \mathcal{X}$ is represented by an n-tuple $X = (x_1, \dots, x_n)$, and $\mathcal{X}$ may be identified with $\mathbf{R}^n$. Expectation with respect to a generic probability measure Q takes the form

$$\mathbf{E}_Q X = \sum_{i=1}^n x_i q_i, \qquad (2)$$

in particular, expectation with respect to the basic probability measure $\mathbf{P}$ equals to

$$\mathbf{E}X = \sum_{i=1}^n x_i p_i. \qquad (3)$$

Risk measure is any real-valued functional $\mu : \mathcal{X} \to \mathbf{R}$ defined on the set of risks $\mathcal{X}$. If a value of a risk measure $\mu(X)$ is completely defined by the distribution function F_X of X for any $X \in \mathcal{X}$, then the risk measure may be thought of as a functional defined on $\mathcal{F}$. We will call such risk measures regular[2].

Recall two concepts of stochastic dominance here. Let $F, G \in \mathcal{F}$ be distribution functions. F dominates G in the sense of first stochastic dominance: $G \le_1 F$, if $F(t) \le G(t)$, $t \in \mathbf{R}$. Note that for $X, Y \in \mathcal{X}$, $X \le Y$ implies $F_X \le_1 F_Y$. Next, for a distribution function $F \in \mathcal{F}$ consider the integral distribution function

$$F^{(2)}(x) = \int_{-\infty}^{x} F(t)\, dt.$$

[1] We say that a distribution function F has a bounded support $[a, b]$, if $-\infty < a < b \le \infty$, $F(x) = 0$ for $x < a$ and $F(x) = 1$ for $x \ge b$.

[2] In [6] such risk measures were called law invariant.

For $F, G \in \mathcal{F}$, F dominates G in the sense of second stochastic dominance[3]: $G \leq_2 F$, if $F^{(2)}(t) \leq G^{(2)}(t)$, $t \in \mathbf{R}$. Note that first and second stochastic dominance constitute partial orderings on $\mathcal{X}$ (or $\mathcal{F}$). In what follows we will call a risk measure $\mu : \mathcal{X} \to \mathbf{R}$ monotone with respect to partial order $\leq$, if $X \leq Y$ implies $\mu(X) \leq \mu(Y)$. The similar concept applies to risk measures of the form $\mu : \mathcal{F} \to \mathbf{R}$.

3 Defining risk measures

Risk measure, as a functional on $\mathcal{X}$ or $\mathcal{F}$, may be defined explicitly or by describing its properties. In either case the result might be a class of functionals, possessing specified properties (axioms) or derived from explicit formula. In the present section we present two ways of axiomatically defining risk measures. The first relies on a concept of acceptable risks, while the second uses a family of functionals to produce a new functional.

Now let us define some properties of risk measures for future using. A risk measure $\mu : \mathcal{X} \to \mathbf{R}$ is called translation invariant if

$$\mu(X + aI) = \mu(X) + a, \quad X \in \mathcal{X}, \ a \in \mathbf{R}. \tag{4}$$

μ is called monotone in value if

$$X \leq Y \Longrightarrow \mu(X) \leq \mu(Y), \quad X, Y \in \mathcal{X}. \tag{5}$$

μ is called sub(super)additive if

$$\mu(X + Y) \leq (\geq) \mu(X) + \mu(Y), \quad X, Y \in \mathcal{X}. \tag{6}$$

μ is called positively homogeneous if

$$\mu(\lambda X) = \lambda \mu(X), \quad X \in \mathcal{X}, \ \lambda \geq 0. \tag{7}$$

Denote $L_+ \subseteq \mathcal{X}$ the nonnegative cone of $\mathcal{X}$, that is, the set of all random variables $X \in \mathcal{X}$ taking nonnegative values with probability 1:

$$L_+ = \{X \in \mathcal{X} : \ \mathbf{P}(\omega \in \Omega : \ X(\omega) \geq 0) = 1\}$$

and $L_- = -L_+$ – the non-positive cone. Denote $\mathcal{A} \subset \mathcal{X}$ a set of risks that we would treat as acceptable, that is, any risk $X \in \mathcal{A}$ may be taken without additional reward. The following assumptions about this set

$$L_+ \subseteq \mathcal{A}, \quad L_- \cap \mathcal{A} = \{0\} \tag{8}$$

[3] Remind that we treat X as gain. If X is treated as loss, using integral of decumulative distribution function $S_X(t) = \mathbf{P}(X > t)$, $t \in \mathbf{R}$ in the definition of the second stochastic dominance is more appropriate.

seem reasonable. The first inclusion in (8) states that any nonnegative risk is acceptable. The second one states that only constant zero risk is acceptable among non-positive risks: if a risk does not bring gain in any state of nature, and brings loss in some states, it cannot be accepted without additional reward.

Now, given any acceptance set $\mathcal{A}$, possessing properties (8), one can define the corresponding risk measure $\mu_{\mathcal{A}} : \mathcal{X} \to \mathbf{R}$ as follows:

$$\mu_{\mathcal{A}}(X) = \sup\{r \in \mathbf{R} : \; X - rI \in \mathcal{A}\}. \tag{9}$$

If X is acceptable, then $\mu_{\mathcal{A}}(X)$ means the maximum certain amount (capital), that may be subtracted from X without leaving the acceptance set. If X is not acceptable ($X \notin \mathcal{A}$), then $\mu_{\mathcal{A}}(X)$ is non-positive, and its absolute value indicates how large certain capital should be added to X for moving X into the acceptance set. In either case $\mu_{\mathcal{A}}(X)$ may be treated as a certainty equivalent for X. Note that the conditions (8) ensure that $\mu_{\mathcal{A}}$ in (9) takes only finite values. Note also that any risk measure defined via (9) is translation invariant and monotone.

Given any monotone, translation invariant, finite risk measure $\mu : \mathcal{X} \to \mathbf{R}$, the corresponding acceptance set $\mathcal{A} = \mathcal{A}_\mu$ is defined as

$$\mathcal{A}_\mu = \{X \in \mathcal{X} : \; \mu(X) \geq 0\}. \tag{10}$$

Next consider building risk measures using a class of predefined functionals. Let Λ be an index set and $\mathcal{M}$ be a set of functionals: $\mathcal{M} = \{\mu_\lambda, \, \lambda \in \Lambda\}$, so that $\mu_\lambda : \mathcal{X} \to \mathbf{R}$ for each $\lambda \in \Lambda$. Then extremal functionals

$$\mu^s_\Lambda(X) = \sup_{\lambda \in \Lambda} \mu_\lambda(X), \quad X \in \mathcal{X} \tag{11}$$

and

$$\mu^i_\Lambda(X) = \inf_{\lambda \in \Lambda} \mu_\lambda(X), \quad X \in \mathcal{X} \tag{12}$$

constitute new risk measures on $\mathcal{X}$. Further, if Λ is endowed with a probability space structure $(\Lambda, \mathcal{C}, S)$, then averaging with respect to the probability measure S also provides a new risk measure

$$\mu^a_\Lambda(X) = \int_\Lambda \mu_\lambda(X) \, dS(\lambda), \quad X \in \mathcal{X}. \tag{13}$$

Extremal operations are often used in conjunction with a class of expectations as basic functionals. In this case each $\lambda \in \Lambda$ represents a probability measure on $(\Omega, \mathcal{B})$, and

$$\mu_\lambda(X) = \mathbf{E}_\lambda X, \quad X \in \mathcal{X}, \quad \lambda \in \Lambda. \tag{14}$$

An example of using (12), (14) is given below in a representation theorem for coherent risk measures in section 5. An example of using the averaging operation (13) for building a distorted probability measure from the class of all VaR measures is presented in section 5.

To complete the section, let us state a proposition on preserving the properties of risk measures under extremal and averaging operations; the proof of the proposition is straightforward.

Proposition 3.1 *If each μ_λ, $\lambda \in \Lambda$ is translation invariant (monotone, positively homogeneous), then μ^s_Λ, μ^i_Λ and μ^a_Λ are also translation invariant (monotone, positively homogeneous). If each μ_λ, $\lambda \in \Lambda$ is sub(super)additive then μ^a_Λ and μ^s_Λ (μ^i_Λ) are also sub(super)additive.*

4 Expected utility measure

Expected utility has been used at least since Daniel Bernoulli constructed his famous St. Petersburg paradox, explaining the flaws of expectation as a certainty equivalent for a risky project. Later the expected utility principle was provided with a solid foundation by John von Neumann and Oscar Morgenstern in their seminal book [7]. The expected utility functional $\rho : \mathcal{X} \to \mathbf{R}$ is defined via a distribution function of an argument, thus being regular:

$$\rho(X) = \rho_U(X) = \mathbf{E}U(X) = \int_{-\infty}^{\infty} U(t)\, dF_X(t) = \int_0^1 U(F_X^{-1}(v))\, dv. \tag{15}$$

Here $U : \mathbf{R} \to \mathbf{R}$ stands for utility function, a parameter of the functional (15).

Denote $U_0(x) = x$, $x \in \mathbf{R}$. Clearly, ρ_{U_0} coincides with expectation functional $\rho_{U_0}(X) = \mathbf{E}X$, $X \in \mathcal{X}$, so expectation is a special case of (15). In general, the expected utility functional may be treated as calculating the expectation of a dollar transform.

Commonly used classes of utility functions are 1) exponential

$$U(t) = (1 - \exp(-\alpha t))/(1 - \exp(-\alpha)),\ \alpha > 0,\ t \in \mathbf{R}, \tag{16}$$

2) power

$$U(t) = t^\alpha,\ t \geq 0,\ \alpha \in (0, 1) \tag{17}$$

and 3) logarithmic

$$U(t) = \ln(1 + \alpha t)/\ln(1 + \alpha),\ t > -1/\alpha,\ \alpha > 0. \tag{18}$$

The last two classes are intended for risks bounded from below, e.g. nonnegative risks. Taking the limit as $\alpha \to 0$ in (16) and (18), and as $\alpha \to 1$ in (17) provides the boundary utility U_0.

When the parameter α satisfies the conditions given in (16) – (18), all these utility functions are increasing and concave, the corresponding expected utility functionals are monotone with respect to first and second stochastic dominance, and exhibit risk aversion in the sense of [9].

When the sample space Ω is finite: $|\Omega| = n$, given the basic probability measure $\mathbf{P} = (p_1, \dots, p_n)$ and a random variable $X = (x_1, \dots, x_n)$, the expected utility functional is calculated as

$$\rho_U(X) = \sum_{k=1}^{n} U(x_k) p_k. \tag{19}$$

The main disadvantage of the expected utility functional is its linearity with respect to a mixture of distributions; given $F, G \in \mathcal{F}$ such that $\mu(F) = \mu(G)$, the following equality holds:

$$\mu(\delta F + (1-\delta)G) = \mu(F) = \mu(G), \ \ \delta \in [0,1]. \tag{20}$$

This means that if two risks F, G are equivalent (in the sense of preference), then their convex combinations are also equivalent to both F and G. In other words, indifference curves of an expected utility functional are straight lines in the space of distributions. Figure 1 demonstrates indifference curves of an expected utility functional in case $|\Omega| = 3$, $X = (1, 2, 3)$, with exponential utility function (16) and the parameter specified in the caption.

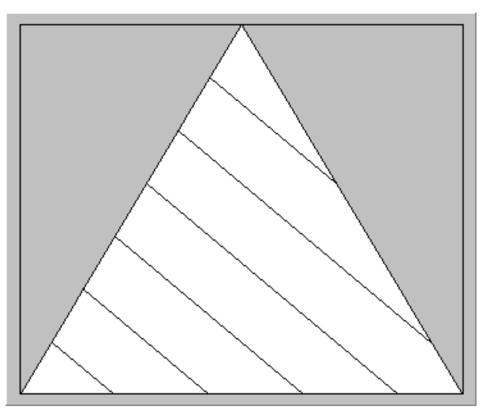
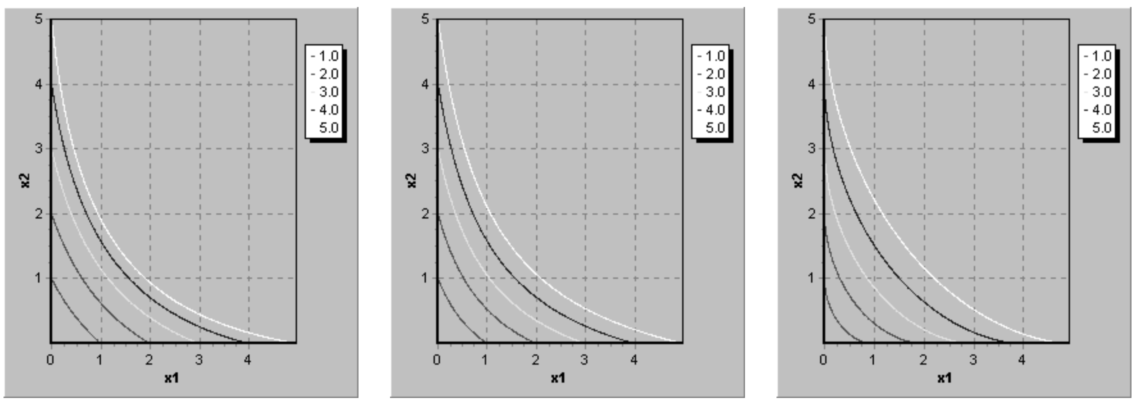


Figure 1: Expected utility indifference curves in the space of distributions $\mathcal{F} = S_3$, $\alpha = 0.7$

As experiments show [4], this feature often differs from real human preferences. Thus the expected utility functional may be regarded as a linear approximation (perhaps, very poor) to a decision-maker's preferences.

Figure 2 exhibits indifference curves of expected utility functionals in the space of random variables $\mathcal{X}$. Here $|\Omega| = 2$, $\mathbf{P} = (1/2, 1/2)$, and the picture shows fragments of indifference curves inside nonnegative cone of $\mathcal{X} = \mathbf{R}^2$. Types of utility functions and parameter values are specified in the caption.



a) exponential, $\alpha = 0.4$ b) logarithmic, $\alpha = 0.9$ c) power, $\alpha = 0.6$

Figure 2: Expected utility indifference curves in the space of random variables $\mathbf{R}^2$

5 The distorted probability measure

The distorted probability measure was introduced in [14] for calculation of an insurance premium for nonnegative risks (losses) and was generalized in [15] to risks taking both positive and negative values:

$$\pi_g(X) = \int_{-\infty}^{0} [g(1 - F_X(t)) - 1]\, dt + \int_{0}^{\infty} g(1 - F_X(t))\, dt. \tag{21}$$

Here $g : [0, 1] \to [0, 1]$ is a distortion function, which is nondecreasing and satisfies $g(0) = 0$, $g(1) = 1$.

A simple algebra provides a representation

$$\pi_g(X) = -\int_0^1 F_X^{-1}(v)\, dg(1 - v), \quad X \in \mathcal{X}, \tag{22}$$

which resembles the last expression for expectation in (1). This allows treatment of π_g as an expectation calculated with a probability transform. Note that in case $g_0(v) = v$, $v \in [0, 1]$ the functional π_{g_0} coincides with expectation: $\pi_{g_0}(X) = \mathbf{E}X$, $X \in \mathcal{X}$. Note also that the functional is regular, so it may be considered as a functional on $\mathcal{F}$:

$$\pi_g(F) = -\int_0^1 F^{-1}(v)\, dg(1 - v), \quad F \in \mathcal{F}. \tag{23}$$

It is clear from (21) that since g is nondecreasing, the functional π_g is monotone in value and with respect to the first stochastic dominance. It is also straightforward that π_g is translation invariant and positively homogeneous. Monotonicity with respect to the second stochastic dominance requires additional assumptions.

Proposition 5.1 *The distorted probability functional π_g is monotone with respect to the second stochastic dominance iff the distortion function g is convex.*

Moreover, convexity of g provides more properties of π_g, stated in the following proposition

Proposition 5.2 *Let g be a convex function. Then the functional π_g is super-additive, concave in value:*

$$\pi_g(\delta X + (1-\delta)Y) \geq \delta\pi_g(X) + (1-\delta)\pi_g(Y)), \quad X, Y \in \mathcal{X}, \ \delta \in [0,1] \tag{24}$$

and convex in distribution:

$$\pi_g(\delta F + (1-\delta)G) \leq \delta\pi_g(F) + (1-\delta)\pi_g(G), \quad F, G \in \mathcal{F}, \ \delta \in [0,1]. \tag{25}$$

Proofs of propositions 5.1 and 5.2 may be found in [8].

Artzner et al [2] have recently introduced a class of risk measures generalizing a distorted probability measure, which they called coherent risk measures. Here we will briefly describe some properties of coherent risk measures that are necessary for what follows. Our definition of coherent risk measures slightly differs from that of [2].

Definition 5.1 *A risk measure $\mu : \mathcal{X} \to \mathbf{R}$ is called coherent, if it is translation invariant, monotone in value, positive homogeneous and superadditive*[4].

The following representation theorem is a reformulation of proposition 4.1 from [2] for the current setting.

Theorem 5.1 *A risk measure $\mu : \mathcal{X} \to \mathbf{R}$ is coherent iff there exists a set of probability measures $\mathcal{Q}$ on $(\Omega, \mathcal{B})$ such that*

$$\mu(X) = \mu_{\mathcal{Q}}(X) = \inf_{Q \in \mathcal{Q}} \mathbf{E}_Q X, \ X \in \mathcal{X}. \tag{26}$$

We will call any set of measures $\mathcal{Q}$, satisfying (26) a generator for the risk measure $\mu_{\mathcal{Q}}$.

[4] In [2] the risk measure $-\mu$ is studied, which is sub-additive.

Remark 5.1 *It can be easily seen that given a set of probability measures $\mathcal{Q}$, representing the risk measure $\mu_{\mathcal{Q}}$, the closed convex hull[5] $Co(\mathcal{Q})$ and any set $\widetilde{\mathcal{Q}}$ in between: $Q \subseteq \widetilde{\mathcal{Q}} \subseteq Co(\mathcal{Q})$ generate the same risk measure $\mu_{\mathcal{Q}} = \mu_{\widetilde{\mathcal{Q}}} = \mu_{Co(Q)}$ by (26). This observation allows choosing different generating sets of measures for a risk measure as appropriate.*

As was mentioned before, the distorted probability measure is positively homogeneous, monotone in value and translation invariant. Proposition 5.2 ensures that the functional is super-additive, provided that the distortion g is convex. This leads to the following proposition.

Proposition 5.3 *A distorted probability measure π_g is coherent, provided that the distortion function g is convex.*

Now consider some examples of convex distortion functions. Power family is given by

$$g(v) = v^{1/\beta}, \quad v \in [0,1],\ \beta \in (0,1]. \tag{27}$$

Dual power functions have the form

$$g(v) = 1 - (1-v)^{\beta}, \quad v \in [0,1],\ \beta \in (0,1]. \tag{28}$$

The exponential family is given by

$$g(v) = (\exp(\beta v) - 1)/(\exp(\beta) - 1), \quad v \in [0,1],\ \beta \in (0,1]. \tag{29}$$

The limiting cases $\beta = 1$ in (27), (28) and $\beta \to 0$ in (29) represent pure expectation $\pi_{g_0}(X) = \mathbf{E}X,\ X \in \mathcal{X}$.

More examples may be found in [14]. The concave distortion functions $\tilde{g}$, presented therein, may be converted to dual convex functions by the transform $g(v) = 1 - \tilde{g}(1 - v),\ v \in [0,1]$.

In case the sample space Ω is finite: $|\Omega| = n$, given the basic probability measure $\mathbf{P} = (p_1, \dots, p_n)$ and a random variable $X = (x_1, \dots, x_n)$ such that $x_1 \le \dots \le x_n$, then the distorted probability functional is calculated as

$$\begin{aligned} \pi_g(X) &= \sum_{k=1}^{n} x_k \left[g\left(\sum_{i=k}^{n} p_i\right) - g\left(\sum_{i=k+1}^{n} p_i\right)\right] & (30)\\ &= \sum_{k=1}^{n} (x_k - x_{k-1}) g\left(\sum_{i=k}^{n} p_i\right), & (31) \end{aligned}$$

[5]Closed convex hull $Co(A)$ of a set A consists of all finite convex combinations of elements of A, that is,

$$Co(A) = \overline{\left\{\sum_{i=1}^{k} \lambda_i a_i,\ a_i \in A,\ \lambda_i \ge 0,\ i = 1, \dots, k,\ \sum_{i=1}^{k} \lambda_i = 1,\ k = 1, 2, \dots \right\}}$$

with the convention that

$$x_0 = 0, \quad \sum_{i=n+1}^{n} p_i = 0. \tag{32}$$

Next let us show that distorted probability measure may be considered as a mixture (in the sense of (13)) of popular value-at-risk measures. For $\lambda \in (0, 1)$ denote

$$\nu_\lambda(X) = F_X^{-1}(\lambda), \quad X \in \mathcal{X} \tag{33}$$

the value-at-risk at level λ.

Theorem 5.2 *Let G be a distribution function on $(0, 1)$ with $G(0) = 0$ and $G(1) = 1$. Consider a mixture of functionals (33) with respect to the distribution G:*

$$\mu(X) = \int_0^1 \nu_\lambda(X)\, dG(\lambda), \quad X \in \mathcal{X}.$$

Then μ coincides with the distorted probability functional π_g with parameter function $g(v) = 1 - G(1 - v),\ v \in [0, 1]$.

Proof. Indeed, substitution of (33) into the expression for μ gives

$$\mu(X) = \int_0^1 F_X^{-1}(\lambda)\, dG(\lambda). \tag{34}$$

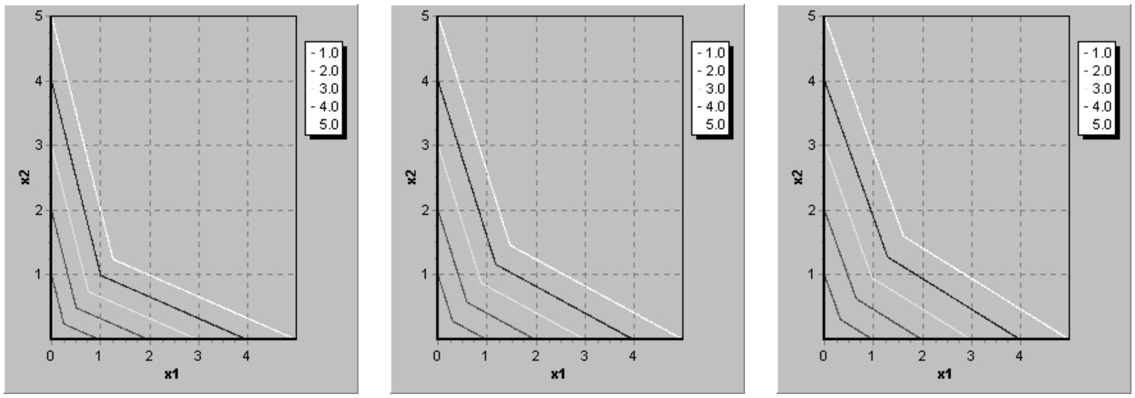
Comparison of (34) with (22) brings the result.

Coherent risk measures as well as their special case, distorted probability measures, are positive homogeneous. This property is often undesirable. Consider a lottery paying a moderate amount a in case of a win, that occurs with probability p (and paying nothing with probability $1 - p$). Let the price of the lottery be b. A risk averse person willing to pay the specified price for this lottery might think that the price $1000b$ for a similar lottery with winning payment $1000a$ is too large.

On the other hand, consider an insurance portfolio consisting of a number of risks, each bringing loss $-a$ with probability p, and insurance premium b. If a single similar risk with loss size $-1000a$ (and the same probability p) is to be added to the portfolio, an insurance company would hardly consider a premium size $1000b$ as sufficient.

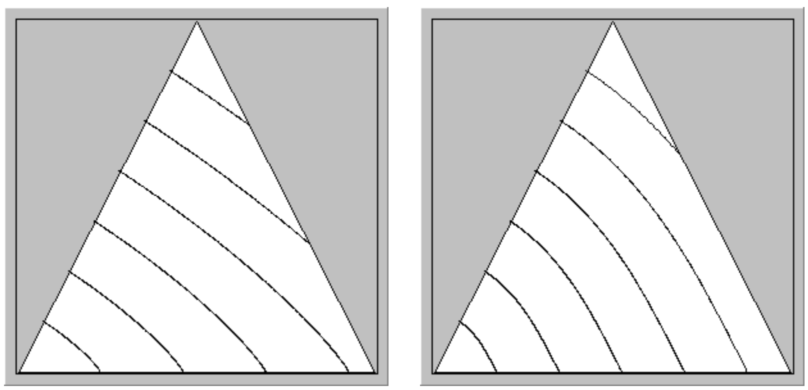
Figure 3 presents indifference curves of a distorted probability functional in the space of random variables in the special case $|\Omega| = 2$, $\mathcal{X} = \mathbf{R}^2$, $\mathbf{P} = (1/2, 1/2)$. Types of distortion functions and their parameters are specified in the caption.

Figure 4 exhibits indifference curves of the distorted probability functional π_g in distributions standard simplex S_3 in the case $|\Omega| = 3$. Here $X = (1, 2, 3)$, and the distortion function g is from the power class (27).



a) power, $\beta = 0.5$ b) dual power, $\beta = 0.5$ c) exponential, $\beta = 1.5$

Figure 3: Distorted probability indifference curves in the space of random variables $\mathbf{R}^2$



a) power, $\beta = 0.7$ b) power, $\beta = 0.3$

Figure 4: Distorted probability indifference curves in the space of distributions S_3

We can now conclude that both expected utility and distorted probability measures possess a sort of linearity, which prevent proper reflection of individual risk aversion. In section 7 we will propose a combined functional, which allows removal of both sorts of linearity, thus enabling a more flexible environment for reflecting individual preferences. The next section will be devoted to deriving a specific form of the representation theorem 5.1 for a distorted probability measure.

6 Representation theorem

Let us consider the representation theorem 5.1 for the special case of a distorted probability measure in more detail. This consideration contains an essential part of the corresponding theorem for combined risk measures as well.

We confine ourselves to the case of finite sample space $|\Omega| = n$ to restrict technicalities.

Because in this case $\mathcal{X} = \mathbf{R}^n$, any risk $X \in \mathcal{X}$ is represented by an n-tuple $(x_1, \ldots, x_n)$, expectation with respect to the basic probability measure $\mathbf{P} = (p_1, \ldots, p_n)$ and generic probability measure $Q = (q_1, \ldots, q_n)$ are expressed as in (3) and (2).

Let us fix a convex distortion function g, then theorem 5.1 guarantees existence of a set of probability measures $\mathcal{Q}$ such that[6]

$$\pi_g(X) = \min_{Q \in \mathcal{Q}} \mathbf{E}_Q X.$$

Thanks to remark 5.1, $\mathcal{Q}$ may be assumed closed and convex to avoid ambiguity. From (22) in case $x_1 \le \cdots \le x_n$ one easily obtains

$$\pi_g(X) = \sum_{k=1}^{n} x_k q_k, \tag{35}$$

where

$$q_k = g(r_k) - g(r_{k+1}), \quad r_k = \sum_{i=k}^{n} p_i, \ k = 1, \ldots, n, \quad r_{n+1} = 0.$$

It is clear that (35) equals the expectation $\mathbf{E}_Q X$ with respect to probability measure $Q = (q_1, \ldots, q_n)$.

If the components of X are not sorted in ascending order, the expression (35) remains valid with different probability measure Q. To calculate the latter consider the set Γ of all $n!$ permutations of the set $\{1, \ldots, n\}$. For a given $X \in \mathcal{X}$ let $\gamma = \gamma_X \in \Gamma$ be a permutation, for which $x_{\gamma(1)} \le x_{\gamma(2)} \le \ldots \le x_{\gamma(n)}$. Denoting

$$q^{\gamma}_{\gamma(k)} = g(r^{\gamma}_k) - g(r^{\gamma}_{k+1}), \ r^{\gamma}_k = \sum_{i=k}^{n} p_{\gamma(i)}, \ k = 1, \ldots, n, \ r^{\gamma}_{n+1} = 0, \tag{36}$$

one concludes that (35) becomes

$$\pi_g(X) = \sum_{k=1}^{n} x_k q^{\gamma}_k \tag{37}$$

with the components of $Q^{\gamma} = (q^{\gamma}_1, \ldots, q^{\gamma}_n)$ calculated via (36). Thus $\pi_g(X) = \mathbf{E}_{Q^{\gamma}} X$ with $\gamma = \gamma_X$ does not exceed expectations of X with respect to measures Q^{γ}, corresponding to other permutations $\gamma \in \Gamma$.

Since components of any $X \in \mathcal{X}$ may be sorted in ascending order by a permutation $\gamma \in \Gamma$, there is a set of at most $n!$ probability measures $\mathcal{Q}_0 = \{Q^{\gamma}, \gamma \in \Gamma\}$ such that $\pi_g(X) = \mathbf{E}_Q X$ for some $Q \in \mathcal{Q}_0$ and

$$\pi_g(X) = \min_{Q \in \mathcal{Q}_0} \mathbf{E}_Q X.$$

Noting that $\mathcal{Q} = Co(\mathcal{Q}_0)$, we conclude the proof of the following theorem.

[6]In finite-dimensional case inf may be substituted with min without trouble.

Theorem 6.1 *Let g be a convex distortion function, $|\Omega| = n$, and Γ be the set of all permutations of $\{1, \dots, n\}$. The distorted probability measure π_g is generated by the set of probability measures $\mathcal{Q}_0 = \{Q^\gamma,\ \gamma \in \Gamma\}$ with components of Q^γ defined by (36). Equivalently, the generator of π_g is the polyhedron $Co(\mathcal{Q}_0)$.*

γ	Q^γ	$\mathbf{E}_{Q^\gamma} X$
(1,2,3)	(7/16,5/16,1/4)	44/16
(1,3,2)	(7/16,1/16,1/2)	36/16
(2,1,3)	(5/16,7/16,1/4)	52/16
(2,3,1)	(1/16,3/4,3/16)	70/16
(3,1,2)	(1/16,7/16,1/2)	60/16
(3,2,1)	(1/16,3/16,3/4)	52/16

Table 1: Generator $\mathcal{Q}_0$ and expectations for $X = (1, 5, 3)$

Consider an example. Let $|\Omega| = 3$, $\mathbf{P} = (1/4, 1/4, 1/2)$, and $g(v) = v^2$, $v \in [0, 1]$. The set of all probability measures constitute the standard simplex in $\mathbf{R}^3$. Table 1 lists all permutations $\gamma \in \Gamma$ and corresponding probability measures Q^γ, which are vertices of the polyhedron $Co(\mathcal{Q}_0)$. Figure 3 presents projections of the generators $Co(\mathcal{Q}_0)$ of π_g onto the plane of the standard simplex for the specified power distortion and dual power distortion $g(v) = 1 - (1 - v)^{0.5}$.

7 Combined risk measure

Since the main flaws of expected utility (dollar transform) and distorted probability (probability transform) risk measures are due to linearity, a natural way of overcoming disadvantages would be simultaneous usage of both transforms. The thought leads to a class of combined functionals

$$\mu_{U,g}(X) = -\int_0^1 U(F_X^{-1}(v))\, dg(1 - v), \tag{38}$$

arising from (1) by applying dollar transform to the integrand and probability transform to the differential part. Here U stands for an utility function as described in section 4, and g stands for distortion function, which was introduced in section 5. This two-parameter class provides much flexibility; it clearly contains all expected utility measures (when $g(v) \equiv v$) and all distorted probability measures (when $U(t) \equiv t$). If U is concave and g is convex, then the functional (38) is concave in value, and may be used in a wide range of decision-making applications for risk averse individuals.

Most of the analysis, presented in sections 4 through 6 directly applies to the combined functional. This risk measure is not positive homogeneous, thus it is not coherent. It is concave in the sense similar to that of [5], and allows similar axiomatic introduction. In contrast to the convex risk measures of [5], the combined risk measure is regular. It is also monotone with respect to first and second stochastic dominance, provided that the parameter functions possess the same necessary properties, as in the partial cases (15) and (21).

Here we will state a representation theorem for (38) and illustrate properties of the functional by its indifference curves, which are shown on figure 4. Parameters in this illustration were adjusted to achieve high similarity of indifference curves with quite different parameter functions to emphasize the flexibility of the functional.

Theorem 7.1 *Let $|\Omega| = n$, the functional (38) be given with concave nondecreasing utility function U and convex nondecreasing distortion function g. Then*

$$\mu_{U,g}(X) = \min_{Q \in \mathcal{Q}_0} \mathbf{E}_Q U(X),$$

where $\mathcal{Q}_0$ is the same as in theorem 6.1.

Proof. Since U is a monotone function, the proof essentially repeats that of theorem 6.1.

It turns out that the utility function U need not be concave to ensure that the combined functional (38) exhibits risk aversion. In what follows we will also use the following class of utility functions

$$U(t) = \begin{cases} \alpha e^t, & t \leq 0, \\ t + \alpha e^{-t}, & t > 0, \end{cases} \tag{39}$$

where the parameter α satisfies $\alpha \in [0, 1]$. Note that each member of the class (39) is piecewise-convex, and even convex for $\alpha \in [0, 0.5]$.

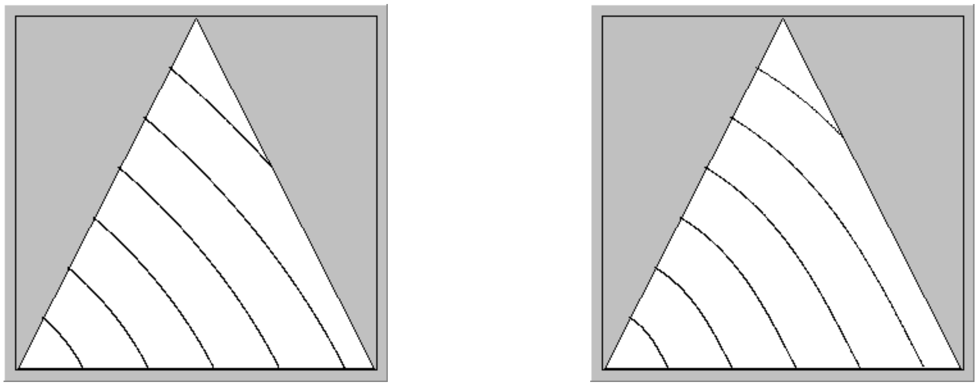
In case the sample space Ω is finite: $|\Omega| = n$, given the basic probability measure $\mathbf{P} = (p_1, \dots, p_n)$ and a random variable $X = (x_1, \dots, x_n)$ such that $x_1 \leq \dots \leq x_n$, the combined functional is calculated as

$$\begin{aligned} \mu_{U,g}(X) &= \sum_{k=1}^{n} U(x_k) \left[g\left(\sum_{i=k}^{n} p_i \right) - g\left(\sum_{i=k+1}^{n} p_i \right) \right] && (40) \\ &= \sum_{k=1}^{n} (U(x_k) - U(x_{k-1})) g\left(\sum_{i=k}^{n} p_i \right), && (41) \end{aligned}$$

with the convention similar to (32):

$$U(x_0) = 0, \quad \sum_{i=n+1}^{n} p_i = 0. \tag{42}$$

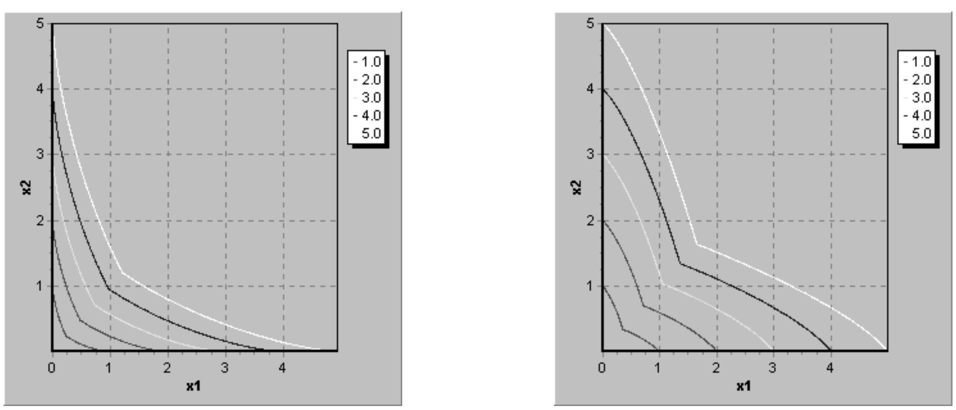
Figure 5 exhibits indifference curves of the combined functional with power distortion function (27), and logarithmic (18) and piecewise-convex (39) utility functions as specified in the captions.



a) logarithmic utility, $\alpha = 2.1$ $\beta = 0.7$ b) piecewise-convex utility $\alpha = 0.8$, $\beta = 0.3$

Figure 5: Indifference curves of combined functional in the space of distributions S_3

Figure 6 presents indifference curves of the combined functional in the space of random variables $\mathbf{R}^2$ with $\mathbf{P} = (1/2, 1/2)$, power distortion function (27) and utility functions specified in captions along with parameters.



a) power utility, $\alpha = 0.7$ $\beta = 0.7$ b) piecewise-convex utility $\alpha = 0.7$, $\beta = 0.5$

Figure 6: Indifference curves of combined functional in the space of random variables $\mathbf{R}^2$

Rabin's paradox. Now consider a paradox that was discovered by Rabin [11], and show how the paradox may be resolved within distorted probability and combined functional frameworks. The paradox is as follows. If a risk averse expected utility maximizer rejects a fair gamble for modest stakes at each initial wealth, then she would reject a gamble with some larger loss and infinitely large gain. Strictly speaking this means the following:

let U be an (increasing, concave) utility function of the decision-maker, $0 < L < G$, denote $R_x(L,G)$ a random variable, taking values $x - L$ and $x + G$ with probability 1/2 each. If the decision-maker rejects such gambles at all initial levels x, which means that $\mathbf{E}U(R_x(L,G)) < U(x)$, $x \in \mathbf{R}$, then there exists a loss $L_0(> L)$ such that the decision-maker would reject any gamble $R_x(L_0, G_0)$, no matter how large is G_0. For example, $L = \$100$, $G = \$125$ imply $L_0 = \$600$, that is, if the decision-makes rejects gambles $R_x(100, 125)$, then she would also reject gambles $R_x(600, G_0)$ for all x, G_0. Another example: rejecting $R_x(100, 110)$ for all x implies rejecting $R_x(1000, G_0)$ for all x, G_0.

The paradox arises because of insufficient flexibility of the expected utility framework for catching individual risk aversion. In what follows we will show that using distorted probability allows avoiding this paradox, while the combined functional provides even more flexibility. First describe the problem formally. Let a decision-maker's preferences be represented by the functional $\mu : \mathcal{X} \to \mathbf{R}$. Given $G > L > 0$, rejecting gambles $R_x(L,G)$ at any initial wealth x means

$$\mu(R_x(L,G)) < \mu(xI), \quad x \in \mathbf{R}. \tag{43}$$

Absence of the paradox means that for any loss $L_0 > L$ there exists a gain $G_0 > G$ such that gambles $R_x(L_0, G_0)$ are not turned down for some (or even all) $x \in \mathbf{R}$. Formally:

$$\mu(R_x(L_0, G_0)) > \mu(xI), \quad \text{for some (all) } x \in \mathbf{R}. \tag{44}$$

Distorted probability framework. First note, that the distorted probability functional π_g is translation invariant, see (4), so turning a gamble $R_x(L,G)$ down for some $x \in \mathbf{R}$ means $\pi_g(R_x(L,G)) < \pi_g(xI)$, which together with (31) gives

$$g\left(\frac{1}{2}\right) < \frac{L}{G+L}\,, \tag{45}$$

implying turning down gambles for all initial wealths x. On the other hand, given a loss L_0, the size of gain G_0 such that gambles $R_x(L_0, G_0)$ are accepted, may be calculated from the inequality $\pi_g(R_x(L_0, G_0)) > \pi_g(xI)$, or, equivalently,

$$g\left(\frac{1}{2}\right) > \frac{L_0}{G_0 + L_0}\,. \tag{46}$$

Combining (45) with (46), one concludes that given L_0, all gambles with $G_0 > GL_0/L$ are accepted provided that the distortion function g satisfies

$$\frac{L_0}{G_0 + L_0} < g\left(\frac{1}{2}\right) < \frac{L}{G+L}\,.$$

In the above example with $L = \$100$, $G = \$125$, $L_0 = \$600$, it is sufficient to choose $G_0 > 750$, say, $G_0 = 800$ and any distortion function g satisfying $3/7 < g(0.5) < 4/9$. This condition is satisfied e.g. by the power distortion function (27) with parameter $\beta = 0.83$.

Combined functional. To simplify formal presentation we will measure gains and losses in hundred dollars. Let us find conditions for inequality $\mu_{U,g}(R_x(L,G)) < \mu_{U,g}(xI)$ to hold for all $x \in \mathbf{R}$. Since $\mu_{U,g}(xI) = U(x)$ and

$$\mu_{U,g}(R_x(L,G)) = U(x-L) + [U(x+G) - U(x-L)]g(1/2), \quad x \in \mathbf{R},$$

we have

$$\frac{U(x) - U(x-L)}{U(x+G) - U(x-L)} > g\left(\frac{1}{2}\right). \tag{47}$$

Denote the fraction on the left $f(x)$, and check the inequality (47) for some parameters U, g. Interestingly, the condition (47) may hold for all $x \in \mathbf{R}$ even for convex utility function U. For example, let

$$U(t) = \begin{cases} 0.5e^t, & t \le 0, \\ t + 0.5e^{-t}, & t > 0 \end{cases} \tag{48}$$

be a member of the class (39) with $\alpha = 0.5$. Then the function f is increasing and

$$f(-\infty) = \lim_{x \to -\infty} f(x) = \frac{1 - e^{-L}}{e^G - e^{-L}}, \quad f(\infty) = \lim_{x \to \infty} f(x) = \frac{L}{G+L}.$$

For example, for $L = 1$, $G = 1.25$ one has $f(-\infty) \approx 0.202$ and $f(\infty) = 4/9 \approx 0.444$. From (47) we see that for $g(v) = v^{1/\beta}$ it suffices that $\beta \le 0.43$.

Now, let parameters of $\mu_{U,g}$ be defined as follows: the utility function U from (48) and

$$g(v) = v^{1/\beta}, \; v \in [0,1] \tag{49}$$

with some $\beta \le 0.43$. Then the decision-maker rejects all gambles $R_x(L,G)$, $x \in \mathbf{R}$. However, she would not turn down gambles with large gains, as was the case for the expected utility maximizer. Indeed, the condition of accepting gambles $R_x(L_0, G_0)$ for all x leads to

$$f_0(x) = \frac{U(x) - U(x-L_0)}{U(x+G_0) - U(x-L_0)} < g\left(\frac{1}{2}\right), \tag{50}$$

similar to (47). The function f_0 is increasing, so (50) is true if

$$\lim_{x \to \infty} f_0(x) = \frac{L_0}{L_0 + G_0} < g\left(\frac{1}{2}\right).$$

Given $L_0 = 6$, the latter condition is satisfied e.g. with $G_0 = 25$ and $\beta = 0.43$. So the decision maker would accept gambles with loss \$600 and gain \$2500 or more. Since U is increasing and tends to ∞ as its argument approaches ∞, it is clear from (50), that similar result holds for any loss size L_0.

8 Conclusion

The paper provides an overview of some methods of building risk measures, which represent human preferences over risky projects. Special attention is paid to expected utility and distorted probability risk measures, that may be treated as applying a dollar transform and a probability transform, respectively, before calculation of an expectation. Since both approaches exhibit a sort of linearity, the combined functional is proposed and studied to some extent.

The main results of the paper are the representation theorem for distorted probability and combined risk measures, and a resolution of Rabin's paradox using nonlinear risk measures. Interesting directions of further research include representing combined risk measures in terms of acceptance sets as in section 3, and in terms of individual preference relation over risks, as was implemented in [8] for some general risk measures. Inverse problems deserve special attention; the greater flexibility of combined risk measures may cause more trouble in selecting the best measure from the class.

Relations among risk measures. Figure 7 presents interrelations among risk measures considered in this paper. Solid arrows mean generalization direction, while dashed arrows depict directions in which generalization is accompanied with some narrowing.

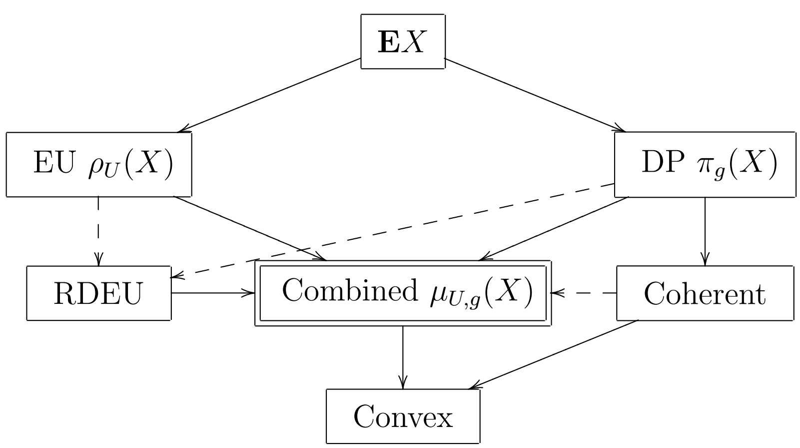


Figure 7: Relations among risk measures

A few illustrations are presented directly in the paper. More may be found in the presentation and the executable, accompanying the paper on the CAS site, and also available from the author's site.

The author would like to express many thanks to Louis Anthony Cox for multiple discussions on expected utility and other decision-making topics.

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