Risk aversion in nonlinear decision-making models

A.A. Novosyolov [1]

Abstract

The risk aversion concept that was introduced by Pratt for expected utility model is being extended in the paper to nonlinear models of individual preferences. Quantitative measure of risk aversion has been proposed and calculated for distorted probability model. Illustrations are provided and a relation of risk aversion to diversification is pointed out.

Keywords and phrases: risk, risk aversion, decision-making, expected utility, distorted probability, diversification.

1 Introduction

The concept of risk aversion was introduced by Pratt [1] in 1964 for expected utility model by von Neumann and Morgenstern [2], and was studied also by K. Arrow [3] and others. Since preference on a set of probability distributions, generated by expected utility functional, turns out to be linear with respect to mixture of distributions, the expected utility model constitutes the first linear approximation to actual human preferences; this fact was also made clear by paradoxes of Rabin [4], Samuelson [5] and Neilson [6]. This invokes the problem of description of risk aversion in nonlinear models of human preferences.

In the current paper risk aversion is being considered in a number of views. First risk aversion is introduced as a characteristic of an abstract preference relation on a set of probability distributions (or a set of random variables). The relation to diversification concept from portfolio analysis is pointed out. Then the concept is defined in terms of representing functionals (risk measures). This allows introducing a quantitative measure of risk aversion as a one-sided analogue of Gateau derivative.

Next risk aversion is being calculated in the distorted probability model, and related illustrations are presented.

Finally some aspects of risk aversion application to decision-making under risk and uncertainty are described, and solving inverse problems of risk theory is discussed; the latter meaning reconstruction individual preferences (or representing risk measures) via observations of actual decision-making.

[1] Institute of computational modeling SB RAS
Academgorodok, Krasnoyarsk, Russia, 660036, e-mail: anov@icm.krasn.ru

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2 Risk aversion

Let $(\Omega, \mathcal{B}, \mathbf{P})$ be a probability space. Consider the set of all almost surely bounded random variables $\mathcal{X} = L^\infty(\Omega, \mathcal{B}, \mathbf{P})$, and a corresponding set $\mathcal{F}$ of distribution functions with bounded support[2]. Elements of $\mathcal{X}$ and $\mathcal{F}$ will be called risks. F_X denotes the distribution function of a random variable $X \in \mathcal{X}$.

A complete transitive binary relation $\preceq$ on $\mathcal{X}$ is called a preference relation. For $X, Y \in \mathcal{X}$ we will treat $X \preceq Y$ as "risk X is not better than risk Y". Asymmetric $\prec$ and symmetric $\sim$ parts of $\preceq$ are strict preference and indifference (equivalence), respectively. Note that $\sim$ is an equivalence relation in the usual sense. If for any $X, Y \in \mathcal{X}$, possessing identical distribution functions $F_X = F_Y$ we have $X \sim Y$, then relation $\preceq$ is called regular; in this case preference relation on $\mathcal{X}$ naturally induces the preference relation on $\mathcal{F}$. In the current paper we will assume that preference relations $\preceq$ are regular and monotone with respect to first stochastic dominance [7]. We will also assume that each equivalence class, generated by $\preceq$, contains exactly one degenerate distribution

$$W_a(x) = \begin{cases} 0, & x < a, \\ 1, & x \geq a, \end{cases} \qquad x \in \mathbf{R},$$

where $a \in \mathbf{R}$ stands for a parameter.

A preference relation $\preceq$ on $\mathcal{X}$ may be represented [7] by a functional (risk measure) $\mu : \mathcal{X} \to \mathbf{R}$ in the sense

$$X \preceq Y \iff \mu(X) \leq \mu(Y), \; X, Y \in \mathcal{X}. \tag{1}$$

If $\preceq$ is regular, the risk measure may be thought of as defined on $\mathcal{F}$.

Denote $I \in \mathcal{X}$ the random variable, that is identically 1: $I(\omega) = 1$, $\omega \in \Omega$ (in particular, for a degenerate distribution one has $W_a = F_{aI}$, $a \in \mathbf{R}$), and introduce a qualitative risk aversion.

Definition 1 *A preference relation $\preceq$ exhibits risk aversion, if for any non-degenerate random variable Δ with zero mean: $\mathbf{E}\Delta = 0$, and any $x \in \mathbf{R}$ the preference $xI + \Delta \prec xI$ holds. In terms of representing risk measure μ (see (1)), risk aversion may be written as $\mu(xI + \Delta) < \mu(xI)$.*

The definition invokes a method for measuring risk aversion. Consider a collection of random variables

$$\mathcal{C} = \{\Delta \in \mathcal{X} : \; \mathbf{E}\Delta = 0, \; \|\Delta\| = 1\}, \tag{2}$$

where

$$\|X\| = a.s. \sup_{\omega \in \Omega} |X(\omega)|$$

[2] A distribution function F on $\mathbf{R}$ has bounded support $[a, b]$, if $-\infty < a < b < \infty$, and $a = \sup\{x : F(x) = 0\}$, $b = \inf\{x : F(x) = 1\}$.

is a $L^\infty(\Omega, \mathcal{B}, \mathbf{P})$ norm. Fix any $\Delta \in \mathcal{C}$. In case of risk aversion, for any $h > 0$ there exists the degenerate (at a point $x - c_{x,\Delta}(h)$) distribution, which is equivalent to $xI + h\Delta$, and $c_{x,\Delta}(h) > 0$ because of monotonicity of $\preceq$ with respect to stochastic dominance. Functional dependence of $c_{x,\Delta}(h)$ on h may be used for characterization of risk aversion at a point x "in direction Δ".

Definition 2 *Let a preference relation $\preceq$ on $\mathcal{F}$ be represented by a risk measure μ. A functional solution $c_{x,\Delta}(h)$ of the equation*

$$\mu(xI + h\Delta) = \mu[(x - c_{x,\Delta}(h))I].$$

is called risk aversion at a point x in direction $\Delta \in \mathcal{C}$.

The limit value

$$c_{x,\Delta}^{(\alpha)} = \lim_{h \searrow 0} c_{x,\Delta}(h)/h^\alpha, \qquad (3)$$

may be used to describe local risk aversion, if it exists for some $\alpha > 0$.

Pratt [1] had shown that linear preference relation, represented by an expected utility functional, exhibits risk aversion if its utility function is strictly concave. He had also calculated the local characteristic of risk aversion of the form (3). Let us presents these results.

Let $U : \mathbf{R} \to \mathbf{R}$ be an utility function. An expected utility functional

$$\rho_U(X) = \mathbf{E}X = \int_{-\infty}^{\infty} U(x)\, dF_X(x), \quad X \in \mathcal{X}$$

is a regular risk measure. If U is increasing, then the functional ρ_U is monotone with respect to stochastic dominance. Risk aversion is equivalent to strict concavity of U. Quantitative expression for risk aversion is presented in the following

Theorem 1 *([1], [8]). Let utility function U be twice continuously differentiable. Then risk aversion in an expected utility model is equal to*

$$c_{x,\Delta}(h) = -\frac{1}{2} h^2 \frac{U''(x)}{U'(x)} + o(h^2), \quad h \to 0.$$

Local risk aversion may be described by

$$c_{x,\Delta}^{(2)} = c_x^{(2)} = -\frac{1}{2} \frac{U''(x)}{U'(x)},$$

which in this case does not depend on direction Δ. The quantity $2c_x^{(2)}$ was called in [1] an absolute risk aversion (in expected utility model).

A similar approach was considered in [8] for nonlinear preference relation, represented by distorted probability functional [9]

$$\pi_g(X) = -\int_0^1 F_X^{-1}(v)\, dg(1-v), \quad X \in \mathcal{X}, \tag{4}$$

where $g:\ [0,1] \to [0,1]$ is a nondecreasing function with $g(0)=0$, $g(1)=1$, which is used to distort the values of the probability measure $\mathbf{P}$. It was shown in [8], that preference relation, corresponding to a functional π_g, exhibits risk aversion if and only if

$$g(v) < v,\ v \in (0,1). \tag{5}$$

The following theorem has been also presented therein.

Theorem 2 *([8]). If the condition (5) holds, then*

$$c_{x,\Delta}(h) = -h\pi_g(\Delta), \quad x \in \mathbf{R},\ h > 0,\ \Delta \in \mathcal{C}. \tag{6}$$

The theorem shows that the ratio $c_{x,\Delta}/h = c^{(1)}_{\Delta} = -\pi_g(\Delta)$ does not depend on $x \in \mathbf{R}$ and $h > 0$, thus $c^{(1)}_{x,\Delta} = c^{(1)}_{\Delta}$ describes both local and global risk aversion in direction Δ. This allows calculating even more informative risk aversion bounds for a functional π_g; they are especially useful in studying recent combined functionals [10]. The bounds will be derived in the next section.

3 Bounds for risk aversion

As has been already pointed out, risk aversion c_Δ in distorted probability model depends only on direction Δ. The current section is devoted to calculation of exact bounds for c_Δ, when Δ in $\mathcal{C}$, in some special cases[3]. Precisely, for a given distortion g the following quantities will be calculated:

$$U_g = \sup_{\Delta \in \mathcal{C}} \pi_g(\Delta), \quad L_g = \inf_{\Delta \in \mathcal{C}} \pi_g(\Delta).$$

In view of (6), these bounds immediately provide bounds for risk aversion as an interval $R_g = [-U_g, -L_g]$. Note that $|\pi_g(X)| \le 1$ for $\|X\| = 1$, and risk aversion implies $-1 \le \pi_g(\Delta) \le 0$ for $\Delta \in \mathcal{C}$, thus $R_g \subseteq [0,1]$. If R_g resides close to 0, then risk aversion is small, while closeness of R_g to 1 corresponds to strong risk aversion. Large length of R_g means significant dependence of risk aversion on direction Δ.

Let Ω be finite: $|\Omega| = n$. Then $\mathcal{X}$ is isomorphic to $\mathbf{R}^n$, each random variable $X \in \mathcal{X}$ may be represented by a vector $(x_1, \dots, x_n)$, the probability measure $\mathbf{P}$ takes the form $(p_1, \dots, , p_n)$, where $p_i \ge 0$, $i = 1, \dots, n$ and $p_1 + \dots + p_n = 1$, and any probability measure

[3] General solution will be reported separately.

Q on $(\Omega, \mathcal{B})$ has the form $Q = (q_1, \dots, q_n)$ with similar conditions for components. In this case the following representation theorem for distorted probability functional has been proven in [10].

Theorem 3 *Let $|\Omega| = n$. Then distorted probability functional (4) may be represented in the form*

$$\pi_g(X) = \inf_{Q \in \mathcal{Q}} \mathbf{E}_Q X, \tag{7}$$

where $\mathbf{E}_Q X = x_1 q_1 + \cdots + x_n q_n$ stands for expectation of a random variable $X \in \mathcal{X}$ with respect to a probability measure Q, the family $\mathcal{Q}$ of probability measures on $(\Omega, \mathcal{B})$ has the form $\mathcal{Q} = \{Q^\gamma,\ \gamma \in \Gamma\}$, Γ is a collection of all $n!$ permutations on the set $\{1, \dots, n\}$, and components of each measure Q^γ are defined by

$$q^\gamma_{\gamma(k)} = g\left(\sum_{i=k}^{n} p_{\gamma(i)}\right) - g\left(\sum_{i=k+1}^{n} p_{\gamma(i)}\right). \tag{8}$$

Now consider examples. Let $n = 2$, then $\mathbf{P} = (p, 1-p)$ with some $p \in (0,1)$, and any probability measure Q on $(\Omega, \mathcal{B})$ has the form $Q = (q, 1-q)$ with some $q \in [0,1]$. Without loss of generality we may assume $p \leq 1/2$. Then the set $\mathcal{C}$ (recall its definition in (2)) contains two points $\Delta^{(1)} = (-1, p/(1-p))$ and $\Delta^{(2)} = (1, -p/(1-p))$. The collection of all permutations of the set $\{1,2\}$ consists of the two elements $\Gamma = \{(1,2),(2,1)\}$, and the corresponding probability measures from (8) are $Q^{(1,2)} = (1 - g(1-p), g(1-p))$, $Q^{(2,1)} = (g(p), 1 - g(p))$. Using the theorem 3, one easily obtains

$$\pi_g\left(\Delta^{(1,2)}\right) = \frac{g(1-p)}{1-p} - 1, \quad \pi_g\left(\Delta^{(2,1)}\right) = \frac{g(p) - p}{1-p}.$$

Let $g(v) = v^3$ and $p = 1/3$, then $L_g = \pi_g\left(\Delta^{(1,2)}\right) = -5/9$ and $U_g = \pi_g\left(\Delta^{(2,1)}\right) = -4/9$, so the range for risk aversion is $R_g = [4/9, 5/9]$. The set $\mathcal{C}$ and the line corresponding to the equation $\mathbf{E}X = 0$ are shown on the figure 1.

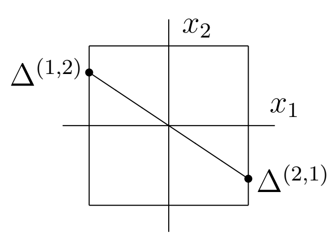

Figure 1: The $\mathcal{C}$ on the square $[-1,1]^2$, $\mathbf{P} = (1/3, 2/3)$

Now let $|\Omega| = 3$. In this case there are 6 permutations on the set $\{1,2,3\}$, and as many probability measures in the family $\mathcal{Q}$. They are presented in the table 1. Let the basic probability measure on $(\Omega, \mathcal{B})$ be $\mathbf{P} = (1/3, 1/3, 1/3)$. The set $\mathcal{C}$ corresponds to a boundary of the hexagon with vertices Δ^γ, $\gamma \in \Gamma$. Figure 2 presents the set $\mathcal{C}$ in the cube $[-1,1]^3$ with coordinates for its vertices and corresponding permutations $\gamma \in \Gamma$. Consider a distortion $g(v) = v^2,\ v \in [0,1]$. It can be easily seen that the minimal value of the functional π_g on the hexagonal boundary is attained at vertices and equals to $-4/9$, while the maximal value is attained at middle points of the sides of the hexagon, and equals to $-1/3$. Thus the range for risk aversion equals to $R_g = [1/3, 4/9]$.

γ	q_1^γ	q_2^γ	q_3^γ
(1,2,3)	$1-g(p_2+p_3)$	$g(p_2+p_3)-g(p_3)$	$g(p_3)$
(1,3,2)	$1-g(p_2+p_3)$	$g(p_2)$	$g(p_2+p_3)-g(p_2)$
(2,1,3)	$g(p_1+p_3)-g(p_3)$	$1-g(p_1+p_3)$	$g(p_3)$
(2,3,1)	$g(p_1)$	$1-g(p_1+p_3)$	$g(p_1+p_3)-g(p_1)$
(3,1,2)	$g(p_1+p_2)-g(p_2)$	$g(p_2)$	$1-g(p_1+p_2)$
(3,2,1)	$g(p_1)$	$g(p_1+p_2)-g(p_1)$	$1-g(p_1+p_2)$

Table 1: Probability measures in the family $\mathcal{Q}$

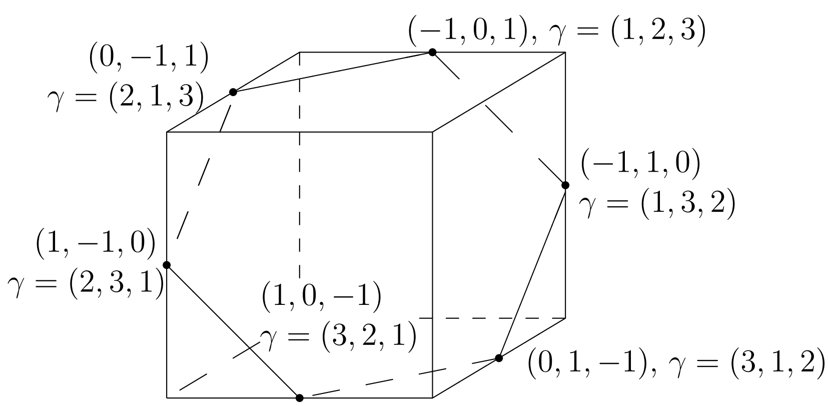

Figure 2: Hexagon $\mathcal{C}$ in the cube $[-1,1]^3$, $\mathbf{P} = (1/3, 1/3, 1/3)$

The example with basic probability measure $\mathbf{P} = (1/4, 1/4, 1/2)$ and the same distortion function $g(v) = v^2,\ v \in [0,1]$ is quite similar. Due to coincidence of two probabilities in $\mathbf{P}$, the hexagon $\mathcal{C}$ becomes the square with vertices $(-1,-1,1)$, $(-1,1,0)$, $(1,1,-1)$, $(1,-1,0)$, and the values of the functional π_g at these vertices are equal to $-1/2$, $-3/8$, $-1/2$ and $-3/8$, respectively. Maximal value of the functional π_g is attained at interior points of the sides of the square, which possess a pair of coinciding coordinates, e.g. $(-1, 1/3, 1/3)$, and is equal to $-1/4$. Thus, the range of risk aversion in this model is equal to $R_g = [1/4, 1/2]$. The set $\mathcal{C}$ and its special points are depicted on figure 3.

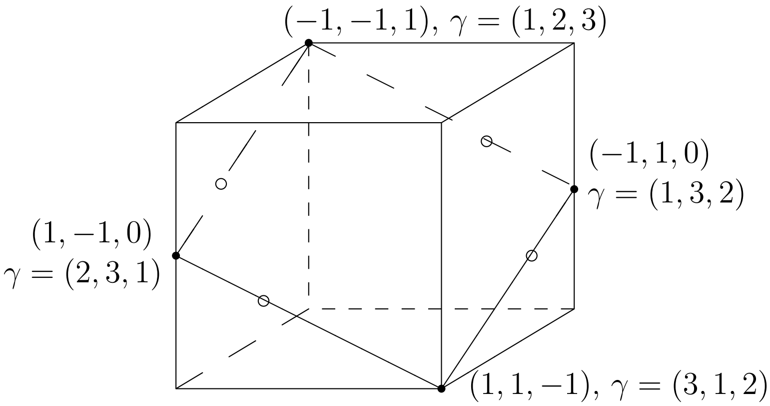

Figure 3: The $\mathcal{C}$ in the cube $[-1,1]^3$, $\mathbf{P}=(1/4,1/4,1/2)$

4 Conclusion

The Arrow – Pratt concept of risk aversion has been extended in the current paper from expected utility models to nonlinear framework. Qualitative risk aversion has been defined for abstract preference relation on a set of risks (real probability distributions), and quantitative risk aversion has been defined for preferences represented by real-valued functionals (risk measures). A range of risk aversion, which is the precise quantitative descriptor of risk aversion, has been introduced for a distorted probability model and calculated in a number of examples. It is worth noting a close relationship of risk aversion with the concept of diversification [12], which is used in portfolio analysis. Deeper study of this relationship deserves much attention.

Quantitative means for measuring risk aversion create wide range of possibilities for studying nonlinear preference models and allow moving solution of inverse problems of risk theory into parametric estimation framework.

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