Measurability of random set of events

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Abstract. *The paper is devoted to establishing unconditional measurability of a random set of events, which follows directly from its structure without any additional requirements.*

Keywords. *Event, set of events, algebra, measurability, terrace.*

1 Introduction

A random element also known as a random set has been studied in the books by O.Yu. Vorobyov [1, 2, 3]. Later eventology was introduced [4], with a specific random set as a central object, precisely — random set of events.

Both abstract random set and random set of events are random elements and as such should be measurable. In the present paper we show that random set of events possesses the measurability property by its status, without additional assumptions.

The second section contains basic concepts and definitions, and the third section presents the proof of the main theorem.

2 Random elements and measurability

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, U be any set, and $\mathcal{A}$ be a σ-algebra of its subsets (note that if U is finite, we may consider algebra of subsets). A random element K on this probability space with values in U is a mapping $K : \Omega \to U$, which is measurable with respect to the pair of σ-algebras $(\mathcal{F}, \mathcal{A})$ in the sense that for each $A \in \mathcal{A}$ we have $K^{-1}(A) \in \mathcal{F}$.

Note for further usage that measurability with respect to a σ-algebra in values set U is *monotone with respect to inclusion*, that is, for any two σ-algebras $\mathcal{A}_1, \mathcal{A}_2$ in U such that $\mathcal{A}_1 \subseteq \mathcal{A}_2$, measurability of K with respect to the pair of σ-algebras $(\mathcal{F}, \mathcal{A}_2)$ implies measurability with respect to the pair of σ-algebras $(\mathcal{F}, \mathcal{A}_1)$. In particular, the most strong measurability is the one with respect to the pair of σ-algebras $(\mathcal{F}, 2^U)$.

A random element is called a random set if its values are subsets of some finite set $\mathcal{X}$, that is, elements of $2^{\mathcal{X}}$. Measurability of a random set is usually defined with respect to the complete algebra of subsets of $2^{\mathcal{X}}$, which is naturally denoted by $2^{(2^{\mathcal{X}})}$.

A random finite set of events is a partial case of a random set. It is defined as follows. Consider a fixed set of events $\mathcal{X} \subset \mathcal{F}$, and define a random element by

$$K(\omega) = \{x \in \mathcal{X} : \omega \in x\}. \tag{1}$$

The expression (1) may be treated as a "random set of events, which had occurred", because the elementary outcome of experiment $\omega \in \Omega$ is mapped to a set of events $X \in \mathcal{X}$, precisely, the set of all events, containing this outcome ω. In other words, those and only those events, which had occurred in this particular experiment.

3 Main theorem

We will consider events in two different spaces of events: in the σ-algebra $\mathcal{F}$, and in the algebra $2^{2^{\mathcal{X}}}$. To distinguish between the two, we will call the latter *hyper-events.*

Theorem 1. *A random set of events K is a measurable mapping from Ω to $2^{\mathcal{X}}$ with respect to the complete algebra of hyper-events $2^{2^{\mathcal{X}}}$ (and, hence, with respect to any algebra of subsets of $2^{\mathcal{X}}$, see monotonicity of measurability with respect to inclusion).*

Proof. We need to show that preimage of any element of hyper-events algebra $2^{2^{\mathcal{X}}}$ is an element of $\mathcal{F}$. Since the algebra $2^{2^{\mathcal{X}}}$ is finite, any of its elements A is a finite union of singletons from $2^{2^{\mathcal{X}}}$,

$$A = \sum_{X \in \Lambda} X, \tag{2}$$

where Λ stands for a set of indices describing the hyper-event A. For example, a hyper-event

$$A_0 = \{\{a\}, \{a, b\}\}$$

is represented as a sum of two singletons

$$\{\{a\}, \{a, b\}\} = \{\{a\}\} + \{\{a, b\}\};$$

here $a, b \in \mathcal{F}$ stand for events in the basic probability space. Preimage of the hyper-event A is by definition the union of preimages of its elements

$$K^{-1}(A) = \sum_{X \in \Lambda} K^{-1}(\{X\}) = \sum_{X \in \Lambda} K^{-1}(X),$$

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for example, for a hyper-event A_0 we get

$$\begin{aligned} K^{-1}(A_0) &= K^{-1}(\{\{a\}\}) + K^{-1}(\{\{a,b\}\}) \\ &= K^{-1}(\{a\}) + K^{-1}(\{a,b\}). \end{aligned}$$

Now everything we need is to note that each expression of the sort $K^{-1}(X)$ for $X \subseteq \mathcal{X}$ is a terrace $ter(X//\mathcal{X})$, defined by a subset of events X in $\mathcal{X}$, that is,

$$ter(X//\mathcal{X}) = \left(\bigcap_{x \in X} x \right) \bigcap \left(\bigcap_{x \in (\mathcal{X} - X)} x^c \right)$$

and a union of such terraces is an element of $\mathcal{F}$, as required.

4 Conclusion

We show that a random set of events is a measurable mapping from a space of elementary events to the space of its values, thus there is no necessity in additional assuming of such measurability.

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