Risk Aversion Concept in Preference Models

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Abstract. *We consider methods of measuring risk aversion in some prefeerence models. In particular, we establish links among risk aversion parameter of exponential expected utility model, required return in the Markowitz problem, and certainty equivalent. We also suggest an idea of measuring risk aversion in coherent risk measures model.*

Keywords. *Preference, risk, expected utility, criteria plane, coherent risk measure, risk measure generator.*

1 Introduction

The risk aversion concept perhaps first appeared in the paper [1] in the framework of expected utility model [2], and was further developed in the book [3]. In exponential expected utility model risk aversion is measured by the parameter α, see (8).

The paper [4] involved a preference model on the criteria plane risk-return; in this model risk aversion may be described by the parameter of required return M, see (2). In the present paper we establish a link between the parameters M, α under fixed normal distribution of returns vector X.

Developing the risk aversion concept from linear expected utility model to non-linear models was started later in the paper [5], and later continued in [6]. A number of related questions were also raised in the recent paper [7].

In the present paper we consider methods of constructing single-parameter sets of preferences, so that the appropriate preference model may be chosen using a single comparison of two risks.

An example of such a preference sets in the classic expected utility preference model is the family based on exponential utility. In distorted probability models the corresponding sets are defined via single-parameter sets of distortion functions [8].

In the present paper we also suggest a method of defining single-parameter families of coherent risk measures via representation of such measures by probability measures families.

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2 Classic models

2.1 Preference, its representation and risk aversion

Complete transitive binary relation on a set $\mathbb{X}$ is called preference on that set [9]. Decision-making under risk is based on a preference relation, which is defined on a set of random variables or distribution functions with finite expectation $EX, X \in \mathbb{X}$. We say that the relation exhibits risk aversion [5], if $X \preceq EX$ for all $X \in \mathbb{X}$. Here we mean that EX denotes not only the corresponding number, but also a random variable which takes the value EX with probability 1.

A preference relation $\preceq$ is represented [9] by a functional $h : \mathbb{X} \to R$, if

$$X \preceq Y \iff h(X) \leq h(Y),\ X, Y \in \mathbb{X}.$$

2.2 Markowitz problem

The preference relation in the Markowitz problem is not complete, because there are incomparable pairs of distributions. However there is an interesting link between Markowitz problem and expected utility theory; in this section we establish some properties of the Markowitz problem.

Consider a random vector X with mean vector $m = \mathbf{E}X$ and covariance matrix $C = \mathbf{E}(XX')$. Assume C is nondegenerate, and denote

$$(x, y) = x'C^{-1}y, \ \ \|x\| = \sqrt{(x, x)}$$

"energetic" scalar product and the corresponding norm in R^n. Using weights vector $y \in R^n$, make up a portfolio $K = y'X$ with expected return $\mathbf{E}K = m'y$ and variance $\mathbf{E}(K - m'y)^2 = y'Cy$. We will call this construct *normal portfolio* for short.

Denote $I \in R^n$ the vector of ones. The Markowitz problem is formulated as follows [4], [10]:

$$\frac{1}{2}\, y'Cy \to \min_{y \in R^n}, \tag{1}$$

subject to

$$m'y = M, \tag{2}$$

$$I'y = 1. \tag{3}$$

Here M stands for the required portfolio return, which is the problem parameter. Denote

$$B = \begin{pmatrix} m & I \end{pmatrix},\ \widetilde{M} = \begin{pmatrix} M & 1 \end{pmatrix}',$$

then constraints (2) — (3) will take the form

$$B'y = \widetilde{M}. \tag{4}$$

Denote also

$$D = B'C^{-1}B = \begin{pmatrix} \|m\|^2 & (m,I) \\ (m,I) & \|I\|^2 \end{pmatrix},$$

$$\Delta = det(D) = \|m\|^2\|I\|^2 - (m,I)^2,$$

and compute

$$D^{-1} = \frac{1}{\Delta}\begin{pmatrix} \|I\|^2 & -(m,I) \\ -(m,I) & \|m\|^2 \end{pmatrix}. \tag{5}$$

The following lemma presents some properties of the Markowitz problem.

Lemma 1. *The problem (1), (4) solution has the form*

$$y = C^{-1}BD^{-1}\widetilde{M},$$

the portfolio variance depends on the required portfolio return M as follows:

$$y'Cy = f(M) = \frac{\|I\|^2M^2 - 2(m,I)M + \|m\|^2}{\Delta}.$$

The minimum variance

$$\sigma_*^2 = 1/\|I\|^2 \tag{6}$$

is attained with

$$M_* = \frac{(m,I)}{\|I\|^2}. \tag{7}$$

For $z > 0$ the equation

$$f'(M) = z$$

has the solution

$$M = M_* + \frac{\Delta}{2\|I\|^2}z.$$

Proof. To solve the problem by Lagrange multipliers method, make up the problem Lagrangian with multipliers $\lambda = (\lambda_1, \lambda_2)'$ as

$$L(y,\lambda) = \frac{1}{2}y'Cy - \lambda'(B'y - \widetilde{M}).$$

Necessary extremum condition $\nabla L_y(y,\lambda) = 0$ gives

$$y = C^{-1}B\lambda.$$

Putting this into (4), we get

$$B'C^{-1}B\lambda = D\lambda = \widetilde{M},$$

which implies

$$\lambda = D^{-1}\widetilde{M}$$

and

$$y = C^{-1}BD^{-1}\widetilde{M},$$

as required.

Direct calculation of portfolio variance gives

$$\begin{aligned} y'Cy &= \widetilde{M}'D^{-1}B'C^{-1}CC^{-1}BD^{-1}\widetilde{M} \\ &= \widetilde{M}'D^{-1}B'C^{-1}BD^{-1}\widetilde{M} \\ &= \widetilde{M}'D^{-1}DD^{-1}\widetilde{M} \\ &= \widetilde{M}'D^{-1}\widetilde{M}, \end{aligned}$$

which together with expression for D^{-1} from (5), implies

$$f(M) = \frac{\|I\|^2M^2 - 2(m,I)M + \|m\|^2}{\Delta},$$

as expected. Calculating the derivative of f with respect to M, we also get the last lemma statement. □

2.3 Expected utility

In expected utility theory preference on the set of real distributions is defined by a functional

$$h(F) = \int_{-\infty}^{\infty} U(x)\,dF(x),$$

where $U : R \to R$ is an utility function. The next theorem establishes conditions for the preference to exhibit risk aversion.

Theorem 1. *The preference defined by the expected utility functional h exhibits risk aversion if and only if the function U is concave.*

Proof. Let $\preceq$ exhibit risk aversion, that is,

$$h(F) \le h(EF), \quad F \in \mathbb{X},$$

where

$$EF = \int_{-\infty}^{\infty} x\,dF(x).$$

For $p \in [0,1]$ consider the family of distribution functions

$$F_{a,b,p}(x) = \begin{cases} 0, & x < a, \\ 1-p, & a \le x < b, \\ 1, & b \le x. \end{cases}$$

For $F = F_{a,b,p}$ we have

$$\begin{aligned} EF &= (1-p)a + pb, \\ EU(F) &= (1-p)U(a) + pU(b), \\ U(EF) &= U((1-p)a + pb), \end{aligned}$$

thus risk aversion

$$U(EF) \ge EU(F), \quad F \in \mathbb{X}$$

implies

$$U((1-p)a+pb) \geq (1-p)U(a)+pU(b),$$

which means concavity of U.

Now let U be concave. Then the classic Jensen's inequality for expectations [11] gives

$$EU(F) \geq U(EF),$$

as required □.

Next calculate the expected utility for a random variable with given normal distribution, using exponential utility function

$$U(x) = 1-\exp(-\alpha x), \; -\infty < x < \infty, \tag{8}$$

which is concave under each $\alpha > 0$. A chart of the function is presented in fig. 1.

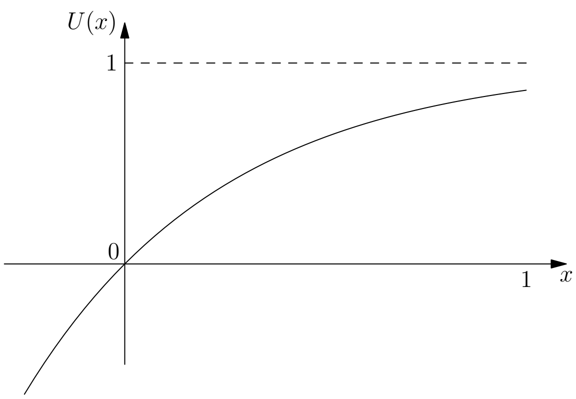

Figure 1: Exponential expected utility

Lemma 2. *Let a random variable X possess normal distribution with mean μ, variance σ^2, and density*

$$\varphi_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), -\infty < x < \infty,$$

and an utility function has the form (8). The the expected utility of X equals

$$\mathbf{E}U(X) = 1-\exp\left(-\alpha\left(\mu-\frac{\alpha\sigma^2}{2}\right)\right).$$

Proof. We have

$$\begin{aligned}
\mathbf{E}U(X) &= \int_{-\infty}^{\infty}(1-\exp(-\alpha x))\,\varphi_{\mu,\sigma}(x)\,dx \\
&= 1-\int_{-\infty}^{\infty}\exp(-\alpha x)\,\varphi_{\mu,\sigma}(x)\,dx \\
&= 1-\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\alpha x-\frac{(x-\mu)^2}{2\sigma^2}\right)dx \\
&= 1-\exp\left(-\alpha\mu+\frac{\alpha^2\sigma^2}{2}\right) \\
&\times \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-(\mu-\alpha\sigma^2))^2}{2\sigma^2}\right)dx \\
&= 1-\exp\left(-\alpha\mu+\frac{\alpha^2\sigma^2}{2}\right),
\end{aligned}$$

as required. □

2.4 Link between Markowitz problem and expected utility theory

It turns out that if a person makes decisions using expected utility functional with exponential utility function with parameter α, then under fixed distribution of the random vector X in the Markowitz problem, the parameter M of the problem describes the risk aversion, and is uniquely related to the parameter α. The relation is established in the following theorem.

Theorem 2. *A person with exponential expected utility $U = U_\alpha$ would choose normal portfolio with*

$$M_\alpha = M_* + \frac{\Delta}{\alpha\|I\|^2}, \;\; \sigma_\alpha^2 = \sigma_*^2 + \frac{\Delta}{\alpha^2\|I\|^2}.$$

An indifference curve passing the point $(M_\alpha, \sigma_\alpha^2)$ is given by the equation

$$\mu = \frac{1}{2\alpha\|I\|^2}\left[\|I\|^2\|m\|^2-(\alpha-(m,I))^2\right]+\frac{\alpha}{2}\sigma^2. \tag{9}$$

Remark 1. *The constant*

$$\mu_{ce} = \frac{1}{2\alpha\|I\|^2}\left[\|I\|^2\|m\|^2-(\alpha-(m,I))^2\right] \tag{10}$$

in (9), corresponding to $\sigma^2 = 0$, is called certainty equivalent for all points in this indifference curve.

Proof. Indifference curves on the criteria plane (μ, σ^2) for the expected utility functional are in fact straight lines with equations

$$\mu - \frac{\alpha\sigma^2}{2} = const;$$

the tangent of he angle between the lines and the μ axis equals $2/\alpha$. At the extreme point an indifference line touches the f function chart, so the tangent equals the derivative of f with respect M at that point. By lemma 1 with $z = 2/\alpha$, we get

$$M_\alpha = M_* + \frac{\Delta}{2\|I\|^2}z = M_* + \frac{\Delta}{\alpha\|I\|^2} \tag{11}$$

and

$$\begin{aligned}
\sigma_\alpha^2 &= f(M_\alpha) = \frac{1}{\Delta}\left(\|I\|^2M_\alpha^2-2(m,I)M_\alpha+\|m\|^2\right) \\
&= \frac{1}{\|I\|^2}+\frac{\Delta}{\alpha^2\|I\|^2} = \sigma_*^2+\frac{\Delta}{\alpha^2\|I\|^2}.
\end{aligned} \tag{12}$$

Substituting the coordinates of $(M_\alpha, \sigma_\alpha^2)$ into the generic indifference line equation, we obtain the required equation (9) with explicit constant expression.□

Thus, given the distribution of the random vector X, the parameters of admissible region of the Markowitz problem are calculated by (6), (7), and required return

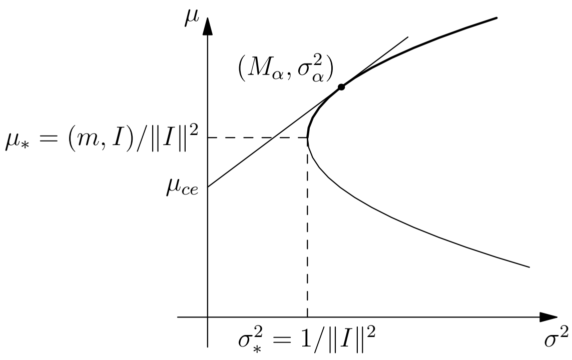

Figure 2: Efficient frontier, extreme indifference line and certainty equivalent of the solution of the Markowitz problem

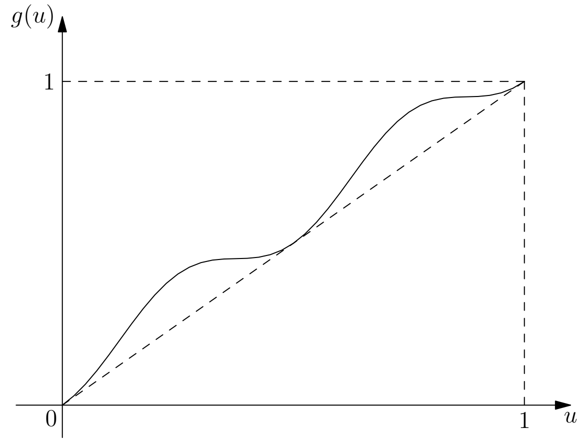

Figure 3: Distortion function

and corresponding risk for a person with risk aversion α are presented in (11), (12).

Thus, assuming some distributional requirements, we may measure risk aversion in the Markowitz problem the same way, as in the expected utility theory.

3 Modern preference models

3.1 Distorted probability model

In this model comparison of distributions is implemented by distorted probability functional [8] (also known as capacity [12]), which maps a real distribution function F to the number

$$\begin{aligned}\pi_g(F) &= -\int_{-\infty}^{0} g\left(F(t)\right)\,dt + \int_0^\infty \left[1 - g(F(t))\right]\,dt \\ &= \int_0^1 F^{-1}(u)\,dg(u),\end{aligned}$$

where $g : [0,1] \to [0,1]$ is a non-decreasing distortion function, satisfying

$$g(0) = 0,\ g(1) = 1.$$

An example chart of distortion function is presented in the fig. 3.

Risk aversion for the functional π_g is equivalent to a simple condition for a function g.

Theorem 3. *Preference defined by a functional π_g exhibits risk aversion if and only if $g(u) \geq u$, $u \in [0,1]$.*

Proof. Let the preference exhibit risk aversion, that is, for any distribution function $F \in \mathbb{X}$ we have $\pi_g(F) \leq \pi_g(EF)$. Since $\pi_g(a) = a$ for any constant a, the assumption implies the inequality

$$\pi_g(F) - EF \leq 0, F \in \mathbb{X}.$$

For $p \in [0,1]$ consider a family of distribution functions

$$F_p(x) = \begin{cases} 0, & x < 0, \\ 1-p, & 0 \leq x < 1, \\ 1, & 1 \leq x \end{cases}.$$

We have

$$F_p^{-1}(u) = \begin{cases} 0, & 0 \leq u < 1-p, \\ 1, & 1-p \leq u \leq 1, \end{cases}$$

so $EF_p = p$ and

$$\pi_g(F_p) = \int_0^1 F_p^{-1}(u)\,dg(u) = 1 - g(1-p),$$

thus

$$\pi_g(F_p) - EF_p = 1 - g(1-p) - p \leq 0,$$

which is equivalent to $g(1-p) \geq (1-p)$, $p \in [0,1]$, as required.

Now let $g(u) \geq u,\ u \in [0,1]$. Denote $\Delta = \pi_g(F) - EF$. Using a well known equality

$$EF = -\int_{-\infty}^{0} F(t)\,dt + \int_0^\infty (1 - F(t))\,dt,$$

we have

$$\begin{aligned}\Delta &= -\int_{-\infty}^{0} g(F(t))\,dt + \int_0^\infty \left[1 - g(F(t))\right]\,dt \\ &+ \int_{-\infty}^{0} F(t)\,dt - \int_0^\infty (1 - F(t))\,dt \\ &= -\int_{-\infty}^{\infty} [g(F(t)) - F(t)]\,dt \leq 0,\end{aligned}$$

as required. □

A number of single-parametric families of distorted functions are presented in [8]. Parameters of these families may be used for describing risk aversion in the corresponding families.

3.2 Coherent risk measures

This class of risk measures was introduced in [13]. We won't get deep into the detailed definition of the class, just recall that each functional of the sort is defined by a convex set of probability measures $\mathcal{Q}$ in the form

$$f(X) = \inf_{Q \in \mathcal{Q}} E_Q X, \ X \in \mathbb{X}.$$

We will call the defining set $\mathcal{Q}$ the *generator* of the coherent risk measure.

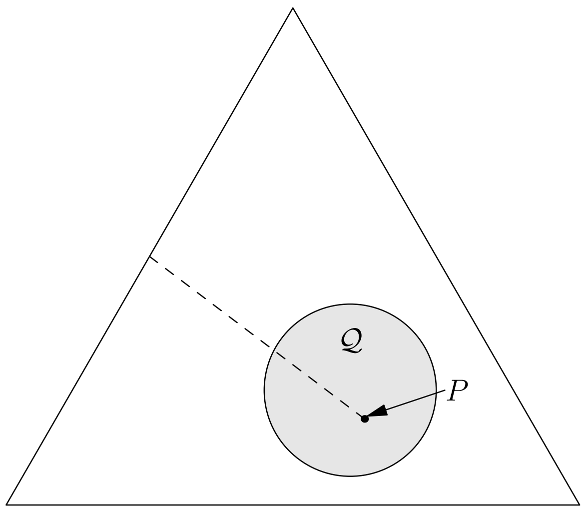

Figure 4: Family of probability measures $\mathcal{Q}$

A qualitative relation between risk aversion on one hand, and form and size of the generator $\mathcal{Q}$ on another hand, may be described as follows. Expanding the generator $\mathcal{Q}$ corresponds to increasing risk aversion. In the limiting case, when the generator fills all the set of probability measures (the triangle in the fig. 4), the risk aversion is maximum; the corresponding coherent risk measure takes the form

$$f_{\mathcal{Q}}(X) = \text{vrai} \inf_{\omega \in \Omega} X(\omega).$$

Squeezing the generator in such a way, that the "true" probability measure P stays within $\mathcal{Q}$, corresponds to decreasing risk aversion. In the limiting case of a single-point generator $\mathcal{Q} = \{\mathcal{P}\}$ we get a risk-neutral person, for which $f(X) = EX$.

It would be interesting to build single-parametric families of generators for building models in the framework of coherent risk measures. We will consider the problem separately.

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