Диверсификация в портфеле
Diversification in a Portfolio

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Abstract

Diversification has been a matter of interest at least since Robinson Crusoe solved his problem of gunpowder allocation. However the concept was not endowed with strict quantitative sense before. In the present paper a measure of diversification is proposed and some of its properties are studied. Possible directions of further research are also mentioned in conclusion.

Keywords: diversification, preference, risk measure, portfolio selection, attitude to risk, risk aversion, stochastic dominance.

Introduction

A concept of diversification is commonly used in a sense of rather vague and uncertain will to "distribute eggs among many baskets". The current paper is devoted to endowing the concept with strict quantitative sense. To be more specific we will consider the concept within portfolio selection framework, though it applies also to other problems of decision-making under uncertainty, resource allocation, and risk management in general.

We first define diversification in a qualitative sense, as a specific property of preference relation on a set of probability distributions. Then we provide the concept with a strict quantitative sense using representation of a preference relation by risk measure.

Qualitative definition

Let $\mathbf{F}$ be a set of real distribution functions such that each distribution function $F \in \mathbf{F}$ represents a result of a decision as in [1]. Let $\prec$ be a preference relation on $\mathbf{F}$, that is, a complete transitive binary relation. For $F, G \in \mathbf{F}$ we interpret the relation $F \prec G$ as "F is not worse than G", or "G is better than or equivalent to F". Two distribution functions $F, G \in \mathbf{F}$ are *equivalent*: $F \sim G$, if both $F \prec G$ and $G \prec F$. This induced equivalence relation as usual provides partition of $\mathbf{F}$ to equivalence classes which form a factor-set. We will call preference relation $\prec$ *regular* if each equivalence class contains exactly one degenerate distribution function $W \in \mathbf{W}$ (see [2]).

For a random variable X denote F_X its distribution function. Any preference relation may be considered on a set $\mathbf{X}$ of random variables as well. A preference relation $\prec$ is called *diversificative* if for any pair $X, Y \in \mathbf{X}$ of equivalent random variables and any number $\lambda \in (0,1)$ the following holds: $X \prec \lambda X + (1-\lambda)Y$. Informally: any mixture of equivalent random variables is at least as good as each of them, perhaps even better. Note that distribution of $\lambda X + (1-\lambda)Y$ depends not only on λ and distributions of X, Y, but on dependence structure of (X, Y) as components of a random vector as well. The above definition suggests that required preference holds for *any* dependence structure of (X,Y). In particular, when X,Y are perfectly positively correlated (e.g. $X = Y$), the mixture coincides with both, and is obviously equivalent to them. Thus the concept of strict diversificativity is relevant; a preference relation is called *strictly diversificative*, if $X \prec \lambda X + (1-\lambda)Y$ holds in the strict sence for any $\lambda \in (0,1)$ and any X,Y, that are not perfectly correlated.

In what follows we will restrict our attention to preference relations that are *concordant with stochastic dominance*, which means that if distribution functions $F,G \in \mathbf{F}$ are such that $F(x) \geq G(x), x \in \mathbf{R}$ (this partial order on $\mathbf{F}$ is denoted by $F \leq_1 G$), then $F \prec G$.

Quantitative definition

Let $\mu : \mathbf{F} \to \mathbf{R}$ be a real-valued functional on $\mathbf{F}$. Remind [1] that it represents preference relation $\prec$ (in other words, it is a risk measure) if

$$F \prec G \Leftrightarrow \mu(F) \leq \mu(G), \quad F,G \in \mathbf{F}. \tag{1}$$

Sometimes it is convenient to think of a functional μ as being defined on $\mathbf{X}$ with clear agreement $\mu(X) = \mu(F_X)$, $X \in \mathbf{X}$. Conditions for existence of representing functional may be found in [1], [3]. Since any functional of the sort induces a preference relation on $\mathbf{F}$, we can identify relation $\prec$ and corresponding functional μ where appropriate. In particular, we can define diversificativity in terms of μ as follows: for any $X,Y \in \mathbf{X}$ such that $\mu(X) = \mu(Y)$, and any number $\lambda \in (0,1)$, the following inequality holds:

$$\mu(\lambda X + (1-\lambda)Y) \geq \mu(X) = \mu(Y).$$

Note that it can be rewritten as

$$\mu(\lambda X + (1-\lambda)Y) \geq \lambda\mu(X) + (1-\lambda)\mu(Y), \tag{2}$$

thus resembling a concavity condition (see [4] for definitions of different types of concavity). The only difference is that the latter is required for arbitrary, rather than only for equivalent, $X,Y \in \mathbf{X}$. However, concave functional necessarily induces diversificative preference. Below we will find out that under some regularity conditions diversificativity of a preference relation $\prec$ is actually equivalent to concavity of representing functional μ.

First introduce a couple of definitions.

Definition 1. A functional $\mu : \mathbf{X} \to \mathbf{R}$ is called *translation invariant* if $\mu(X + a) = \mu(X) + a$ for any $X \in \mathbf{X}$ and any $a \in \mathbf{R}$.

Definition 2. A functional $\mu : \mathbf{X} \to \mathbf{R}$ is called *positively homogeneous* if $\mu(aX) = a\mu(X)$ for any $X \in \mathbf{X}$ and any $a \geq 0$.

Now we can state the

Theorem 1. Let risk measure $\mu : \mathbf{X} \to \mathbf{R}$ be translation invariant. Then it is diversificative if and only if it is concave.

Note that representing functional μ in (1) for a preference relation $\prec$ is not unique; any functional $\nu : \mathbf{F} \to \mathbf{R}$ satisfying

$$\nu(F) = f(\mu(F)), \quad F \in \mathbf{F}, \tag{3}$$

with some strictly increasing real function f, represents the same preference relation; the inverse is also true. We will call functionals μ,ν, satisfying (3), *equivalent*: $\mu \sim \nu$. It turns out that under some regularity conditions essentially all risk measures (representing functionals) are translation invariant, thus diversificativity of a functional is actually equivalent to its concavity. This statement is made rigorous by the

Theorem 2. Let a preference relation $\prec$ be regular and concordant with stochastic dominance. Then there exists the representing functional μ such that $\mu(W_a) = a$, $a \in \mathbf{R}$. Moreover, if $\prec$ is strictly diversificative, then μ is almost strictly concave, which means that

$$\mu(\lambda X + (1-\lambda)Y) > \lambda\mu(X) + (1-\lambda)\mu(Y),$$

for all $\lambda \in (0,1)$ and all $X, Y \in \mathbf{X}$ that are *not* perfectly correlated.

This theorem in particular means that if a non-concave functional represents a diversificative preference relation, then it may be corrected (by a strictly increasing transform) in such a way, that the new functional would be concave and represent the same preference relation.

In what follows let us restrict consideration to regular preferences, concordant with stochastic dominance, and introduce a quantative measure of diversification. Equation (2) provokes the measure of the form $\mu(\lambda X + (1-\lambda)Y) - \lambda\mu(X) + (1-\lambda)\mu(Y)$. We will define the measure in question for a general portfolio setting. Let $(X_1,..., X_n)$ be a random vector representing returns of n market tools or other investment instruments. Denote

$$\mathbf{S}_n = \left\{ y = (y_1,.., y_n) \in \mathbf{R}^n \mid y_1 \geq 0,..., y_n \geq 0, \sum_{i=1}^{n} y_i = 1 \right\}$$

the standard simplex of $\mathbf{R}^n$; its elements y are interpreted as weights of instruments in a portfolio. Given a vector of weights $y \in \mathbf{S}_n$, diversification of the corresponding portfolio $X^y = y_1 X_1 + ... + y_n X_n$ is measured by

$$D(y) = \mu\left(\sum_{i=1}^{n} y_i X_i \right) - \sum_{i=1}^{n} y_i \mu(X_i).$$

Let y^j be j-th ort of $\mathbf{R}^n$, a vertex of $\mathbf{S}_n$. Clearly the measure D is nonnegative, and $D(y^j) = 0$, $j \in N = \{1,..., n\}$. For a subset $J \subseteq N$ denote S_J the convex hull of $\{y^j, j \in J\}$; the latter is a boundary simplex of $\mathbf{S}_n$ when $J \neq N$, and coincides with $\mathbf{S}_n$ when $J = N$. The following theorem describes properties of D in terms of properties of preference and the distribution of $(X_1,...,X_n)$.

Theorem 3. Let preference relation $\prec$ be strictly diversificative. If a subset $J \subseteq N$ is such that $\{X_j, j \in J\}$ are pairwise perfectly correlated, then $D(y) = 0$, $y \in S_J$. If $y \in \mathbf{S}_n$ cannot be represented as a convex combination of pairwise perfectly correlated components of $(X_1,...,X_n)$, then $D(y) > 0$. In particular, if all components of $(X_1,...,X_n)$ are perfectly correlated, then $D(y) \equiv 0$ on $\mathbf{S}_n$, and if there are no perfectly correlated components, then $D(y) > 0$ everywhere on $\mathbf{S}_n$ except for its vertices.

The case when all components of $(X_1,...,X_n)$ are perfectly correlated corresponds to completely degenerate distribution of the random vector. In case of non-degenerate distribution there exists a point $y^* \in \mathbf{S}_n$ where diversification measure D reaches its maximum. This point may be considered as a best diversified portfolio with respect to the underlying preference relation $\prec$.

Conclusion

Diversification has been a matter of interest for long time, though mostly in a qualitative sence. The present paper provides an attempt to supply the concept with strict quantitative description in a portfolio setting using individual preference relation on a set of probability distributions. A very interesting point of further research is to find out how the diversification measure may be used for characterization of risk aversion of an individual. Another point of interest is whether best diversified portfolio may be considered as optimal in the sense of portfolio selection problem. Deeper studying of interrelations between diversification measure and dependence structure of the underlying distribution is also worth attention. Results of this research are to be reported later in a seperate paper.

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