RISK AVERSION IN THE SMALL WITH APPLICATIONS TO PORTFOLIO ANALYSIS

Arcady Novosyolov

Institute of Computational Modelling SB RAS,
Academgorodok, Krasnoyarsk, Russia, 660036,
anov@icm.krasn.ru

Abstract

The concept of risk aversion was introduced by Pratt in 1964 for expected utility framework, and has attracted much attention as an important quantitative characteristic of individual attitude to risk. The current paper extends the concept to other decision-making frameworks, and presents exact quantitative results for distorted probability functional. Applications of the concept to portfolio analysis, and its relation with diversification concept are also being studied.

Keywords

Decision-making, risk, individual preferences, risk measure, risk aversion, expected utility, distorted probability, portfolio, diversification

Introduction

The concept of risk aversion was introduced in [1] for expected utility framework [2]. It was recognized as a significant tool for describing individual attitude to risk, and has been paid much attention in the literature. A number of features of the concept have been studied, including dependence on initial wealth; the resulting classification of decision-makers to those with DARA, CARA and IARA (which mean decreasing, constant and increasing absolute risk aversion) has been established.

A similar concept for nonlinear decision-making models was recently introduced in [3]. The quantitative risk aversion has been calculated for a number of examples therein, however bounds for possible values of risk aversion remain unknown.

The present paper continues developing the concept for distorted probability risk measures. It presents definition of risk aversion in terms of preference relation and in terms of representing risk measure. Then the strict quantitative concept is being defined as a range of values of the risk measure over a unit sphere in appropriate linear normed space. The quantity appears to be calculable, and thus useful for real decision-making problems. Finally an application of risk aversion concept to a simple portfolio selection problem is described.

Risk aversion

Let $(\Omega, \mathbf{B}, \mathbf{P})$ be a probability space, $\mathbf{X}$ be the set L^∞ of almost surely bounded random variables X, and $\mathbf{F}$ be the set of their probability distribution functions on the real line $\mathbf{R}$. Denote $I \in \mathbf{X}$ the constant random variable $I(\omega) \equiv 1$ with degenerate distribution. Let $\prec=$ be a preference relation on $\mathbf{F}$, that is, a complete transitive relation, which is treated as follows: $F \prec= G$ means that the distribution $G \in \mathbf{F}$ is at least as good as the distribution $F \in \mathbf{F}$, perhaps even better. We will also use the notation $F \prec G$ and $F \sim G$ for asymmetric and symmetric parts of $\prec=$, respectively, and denote $\prec=$ the induced preference relation on $\mathbf{X}$.

As it was shown in [4], under some mild assumptions a preference relation $\prec=$ may be represented by a real functional (risk measure) $\mu : \mathbf{F} \to \mathbf{R}$ in the sense

$$F \prec= G \quad \Leftrightarrow \quad \mu(F) \le \mu(G), \quad F, G \in \mathbf{F}. \tag{1}$$

Throughout the paper we will assume that preference is regular [3], id est, each equivalence class contains exactly one degenerate distribution, and the preference is monotone with respect to the first stochastic dominance, in particular $\mu(aI) < \mu(bI)$ for $a < b$.

Denote $\mathbf{F}_0$ the class of all nondegenerate distribution functions $F \in \mathbf{F}$ with zero mean: $\mathbf{E}F = 0$, and $\mathbf{X}_0$ the corresponding class of nondegenerate random variables $X \in \mathbf{X}$ with $\mathbf{E}X = 0$. The concept of risk aversion in terms of preference relation is defined as follows.

Definition 1. A preference relation $\prec=$ on $\mathbf{F}$ exhibits risk aversion, if $xI + \Delta \prec xI$ for any $\Delta \in \mathbf{X}_0$ and any $x \in \mathbf{R}$.

In terms of representing risk measure μ this means $\mu(xI + \Delta) < \mu(xI)$. In other words, adding a zero mean “variation” leads to less preferable distributions.

Now let us supply risk aversion with strict quantitative sense. Consider some norm $\| \cdot \|$ in $\mathbf{X}$ and denote

$$\mathbf{C} = \{X \in \mathbf{X}_0 : \| X \| = 1\}$$

the intersection of $\mathbf{X}_0$ with the unit sphere of $\mathbf{X}$; elements of $\mathbf{C}$ may be thought of as normalized variations. Since $xI + h\Delta \prec xI$ for all $x \in \mathbf{R}$, $h > 0$ and $\Delta \in \mathbf{C}$, there exists the unique $c \in \mathbf{R}$ such that

$$xI + h\Delta \sim (x - c)I. \tag{2}$$

The solution of (2) clearly depends on x, h and Δ, so we will denote it $c_x(h, \Delta)$. In the paper [1] this quantity was called risk premium; we will call it quantitative risk aversion or simply risk aversion here.

Let preference relation satisfy the axioms of von Neumann and Morgenstern [2], or, equivalently, let it be represented by an expected utility functional

$$\rho_U(F) = \int_{-\infty}^{\infty} U(x) dF(x), \qquad F \in \mathbf{F}. \tag{3}$$

Pratt [1] has shown (see also [3]), that in this case risk aversion possesses the following local representation

$$c_x(h, \Delta) = -\frac{1}{2} h^2 E\Delta^2 \frac{U''(x)}{U'(x)} + o(h^2). \tag{4}$$

The quantity $a(x) = -U''(x)/U'(x)$ describes influence of individual preferences on risk aversion $c_x(h, \Delta)$ (it was called absolute risk aversion in [1]). It depends only on “initial capital” x. Choosing Euclidean norm $\| X \| = \mathbf{E}X^2$ in $\mathbf{X}$, one concludes that risk aversion (locally) does not depend on the direction of variation Δ. Dependence on h is of the second order, in particular, decision-maker is asymptotically risk-neutral as $h \to 0$.

Now consider preference relations induced by distorted probability functional [5]

$$\pi_g(F) = -\int_0^1 F^{-1}(v)dg(1-v), \qquad F \in \mathbf{F} \tag{5}$$

Here $g:[0,1] \to [0,1]$ stands for a distortion function, which is increasing, and satisfies $g(0)=0$, $g(1)=1$. This preference relation exhibits risk aversion if the distortion function satisfies $g(v) < v$, $v \in [0,1]$ [3]. In this case risk aversion takes the form

$$c_x(h,\Delta) = c(h,\Delta) = -h\pi_g(\Delta). \tag{6}$$

where values of π_g on $\mathbf{X}$ are calculated via obvious mapping $\pi_g(X) = \pi_g(F_X)$, $X \in \mathbf{X}$, and F_X stands for a distribution function of X.

It clearly does not depend on "initial capital" x. It also differs from the case of expected utility (4) by the order of dependence on h: as variation gets small, the decision-maker does not approach risk-neutrality. Finally, as it will become clear later, risk aversion significantly depends on the direction of variation Δ, thus catching the functional form of the distribution of the variation.

Calculation of risk aversion

Let us calculate risk aversion (6) for preferences induced by a distorted probability functional (5), using the L^∞ norm in $\mathbf{X}$: $\| X \| = ess\sup_{\omega\in\Omega} X(\omega)$.

Theorem 1. *Let the distortion function g be convex and continuous at $v=1$. Then*

$$-1 + g(1/2) \le \pi_g(\Delta) \le 0, \qquad \Delta \in \mathbf{C}, \tag{7}$$

where both upper and lower bounds are exact.

The upper bound is easily verified. Indeed, let $X_p \in \mathbf{C}$ be a family of random variables with distributions $\mathbf{P}(X_p = -1) = \mathbf{P}(X_p = 1) = p$, $\mathbf{P}(X_p = 0) = 1-2p$, $p \in (0,1/2)$. Then $\pi_g(X_p) = g(p) + g(1-p) - 1 \to 0$ as $p \to 0$. Construction of the lower bound is more complicated, and will be published separately.

Note that if the basic probability measure $\mathbf{P}$ possesses atoms, then the bounds may occur tighter than in (7). Consider a finite sample space Ω with $|\Omega| = 3$ and the uniform probability measure $\mathbf{P} = (1/3, 1/3, 1/3)$, and let $g(v) = v^2$, $v \in (0,1)$. As it was shown in [6], the bounds take the form $-4/9 \le \pi_g(\Delta) \le -1/3$, $\Delta \in \mathbf{C}$ instead of (7), which would produce $[-3/4, 0]$ as the interval for possible values of $\pi_g(\Delta)$. The variations set $\mathbf{C}$ in this example is represented by the boundary of the hexagon presented on figure 1. The minimal value $-4/9$ is attained at the vertices of the hexagon, while the maximal value $-1/3$ is attained at middle points of the sides of the hexagon.

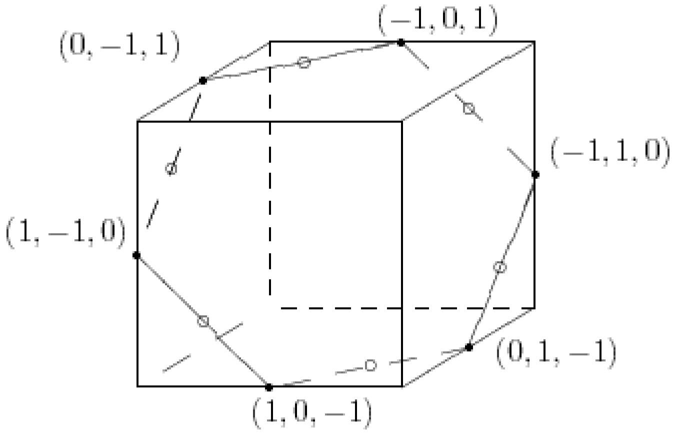

Fig. 1. Hexagon $\mathbf{C}$ in the cube $[-1,1]^3$ under uniform probability measure

Portfolio selection

Now consider some implications of risk aversion to portfolio analysis. Let investor's preference over probability distributions be represented by a distorted probability functional (5). Consider constructing portfolio using a riskless asset with rate of return r (risk-free rate) and a risky asset with random rate of return $Z \in L^\infty$ with $m = \mathbf{E}Z$. In this case a portfolio rate of return is calculated as $X = yr + (1-y)Z$, where y is a weight of the riskless asset in a portfolio. Denote $a = \| Z - m \|_\infty$ and $\Delta = (Z - m)/a$. Clearly, $Z = m + a\Delta$, and $\Delta \in \mathbf{C}$. Hence the value of the distorted probability functional for a portfolio return X equals

$$\pi_g(X) = m + a\pi_g(\Delta) + y[r - m - a\pi_g(\Delta)]. \qquad (8)$$

Combining (8) with assertion of the theorem 1, one can conclude that investors with modest risk aversion (which means $g(1/2) > 1 - (m - r)/a$)) would definitely include only the risky asset in their portfolio. For more risk averse investors the decision requires careful calculation of $\pi_g(\Delta)$.

Conclusion

Risk aversion in nonlinear preference models appears significantly different from its counterparty in the linear (expected utility) world. Under small deviations from certainty, a decision-maker is no longer asymptotically risk-neutral, as it was the case within the linear framework. Because of positive homogeneousness of distorted probability measure, risk aversion in this model does not depend on initial wealth, which apparently would not be the case in more complex nonlinear models, for example, for recently introduced combined functionals [7].

Possible directions of further research of risk aversion include its calculation in other norms in $\mathbf{X}$, e.g. those of L^p, $1 \le p < \infty$, and establishing exact bounds of risk aversion for a number of classes of preference, including those induced by combined functionals. Applications of risk aversion in portfolio analysis are also worth further studying; in particular a problem of great interest is that of interrelation between risk aversion and diversification.

References

1. J. Pratt. Risk Aversion in the Small and in the Large. *Econometrica*. **32** (1964), 122-136.
2. J. von Neumann, O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton: Princeton University Press, 1953.
3. A. Novosyolov. Risk Aversion: Qualitative Approach and Quantitative Estimates. *Automation and Remote Control*, 2003, to appear.
4. A. Novosyolov. *Mathematical Modelling of Financial Risks: Measuring Theory*. Novosibirsk: Nauka, 2001.
5. S. Wang. Premium Calculation by Transforming the Layer Premium Density. *ASTIN Bulletin*, **26** (1996), p. 71-92.
6. A. Novosyolov. Risk Aversion in Nonlinear Decision-Making Models. *Proceedings of the 3rd All-Russian conference "Financial and Actuarial Mathematics and Related Topics"*, Krasnoyarsk, 2003.
7. A. Novosyolov. Combined functionals as risk measures. *Proceedings of the Bowles Symposium "Fair Valuation of Contingent Claims and Benchmark Cost of Capital"*, Atlanta GA, 2003, http://www.casact.org/coneduc/specsem/sp2003/papers/
8. A. Novosyolov. Inverse problems of Risk Theory and Characteristic Classes of Distributions. *Proceedings of the 3rd International Scientific School "Modelling and Analysis of Safety and Risk in Complex Systems"*, St.Petersburg, 2003, this volume.