HIGHER ORDER STOCHASTIC DOMINANCE IN OPTION PRICING AND INSURANCE
СТОХАСТИЧЕСКОЕ ДОМИНИРОВАНИЕ ВЫСШИХ ПОРЯДКОВ
В ОЦЕНИВАНИИ ОПЦИОНОВ И СТРАХОВАНИИ

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The paper contains derivation of integral and asymptotic representations for complementary distribution functions. A few examples illustrate direct and dual second order stochastic dominance in terms of insurance and option pricing.

Key words: integral distribution function, second order stochastic dominance, insurance, franchise, reinsurance, option pricing

## Introduction

A partial ordering on a set of probability distributions known as stochastic dominance plays a significant role in studying individual preferences in risk theory. The concept is described in detail in the book [1]. In [2] some generalizations of the concept were described, their properties were studied; the paper also presented some integral representations for integral distribution functions.

First order stochastic dominance has the clear sense. Sense of higher order stochastic dominance is not that clear. This paper interprets direct and dual second order stochastic dominance in terms of pure premium for risk transfer of several types: franchise insurance, reinsurance, European option pricing.

In the next section basic concepts are introduced, integral distribution functions are defined, integral and asymptotic representation for complementary distribution functions are derived; the latter are analogues of representations of direct distribution functions. In the last section direct and dual stochastic dominance are defined, and a few examples of financial risks are presented with interpretation of second order stochastic dominance.

## Basic concepts and notation

Consider a measurable space $(\mathbf{R},\mathbf{B})$, where $\mathbf{R}$ is a set of real numbers, and $\mathbf{B}$ is its Borel $\sigma$-algebra. A set function $P:\mathbf{B}\to\mathbf{R}$ is called a *probability measure* (*probability distribution*) if it is $\sigma$-additive, takes non-negative values, and $P(\mathbf{R})=1$. Denote $\mathbf{P}$ the set of all probability distributions on $(\mathbf{R},\mathbf{B})$. A probability distribution $P\in\mathbf{P}$ may be described by a *distribution function* $F=F_P$, which is defined by $F_P(x)=P((-\infty,x])$, $x\in\mathbf{R}$, or by a *complementary distribution function* $S_P(x)=1-F_P(x)=P((x,\infty))$, $x\in\mathbf{R}$.

For a distribution function $F$ let us define integral distribution functions $F^{(k)}$, $k=1,2,\dots$ by $F^{(1)}=F$ and

$$F^{(k+1)}(x)=\int_{-\infty}^{x}F^{(k)}(t)dt, \qquad k=1,2,\dots, \qquad x\in\mathbf{R}.$$

For a complementary distribution function $S$ its integral distribution functions are defined similarly: $S^{(1)}=S$, and

$$S^{(k+1)}(x)=\int_{x}^{\infty}S^{(k)}(t)dt, \qquad k=1,2,\dots, \qquad x\in\mathbf{R}.$$

Graphic presentation of integral distribution functions of order 2 is depicted in figure 1.

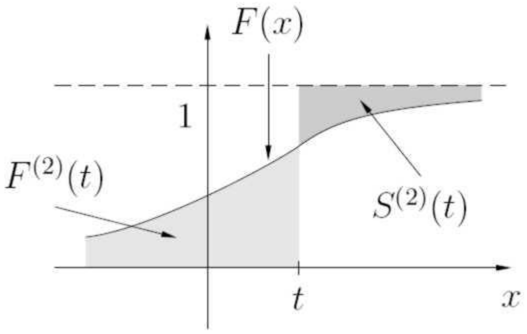



Figure 1. Integral distribution functions, $k=2$

Denote $\mu_k^P$ the $k$-th moment of a distribution $P \in \mathbf{P}$:

$$\mu_k^P = \int_{-\infty}^{\infty} x^k dF_P(x), \qquad k = 1,2,\ldots$$

It is known [2] that existence of moments up to order $k$ ensures existence of integral distribution functions $F^{(k+1)}$ and $S^{(k+1)}$, and the representations

$$F^{(k+1)}(x) = \frac{1}{k!}\int_{-\infty}^{x}(x-t)^k dF(t), \qquad \lim_{x\to\infty}\left(F^{(k+1)}(x) - a_k(x)\right) = 0 \tag{1}$$

where

$$a_k(x) = \frac{1}{k!}\sum_{i=1}^{k}(-1)^i C_k^i x^{k-i}\mu_i, \quad C_k^i = \frac{k!}{i!(k-i)!},$$

that is, $a_k$ is a polynomial of order $k$ with top coefficient $1/k!$, here $C_k^i$ are binomial coefficients.

Representations for complementary distribution functions may be obtained similarly:

$$S^{(k+1)}(x) = \frac{1}{k!}\int_{x}^{\infty}(t-x)^k dF(t), \qquad \lim_{x\to-\infty}\left(S^{(k+1)}(x) - |a_k(x)|\right) = 0. \tag{2}$$

In particular, we have $F^{(2)}(x) \sim x - \mu_1$ as $x \to \infty$, and $S^{(2)}(x) \sim -x + \mu_1$ as $x \to -\infty$.

Consider examples of integral distribution functions. Denote $W_a$ a probability distribution that is degenerate at a point $a \in \mathbf{R}$. The corresponding distribution function $F_{W_a}$ and complementary distribution function $S_{W_a}$ have the form

$$F_{W_a}(x)=\begin{cases}0, & x<a\\ 1, & x\ge a\end{cases}, \qquad S_{W_a}(x)=\begin{cases}1, & x<a\\ 0, & x\ge a\end{cases}$$

Moments of this distribution are equal to $\mu_k = a^k$, $k=1,2,\ldots$, and some integral distribution functions for $W_a$ are

$$F^{(2)}_{W_a}=\begin{cases}0, & x<a\\ x-a, & x\ge a\end{cases}, \qquad S^{(2)}_{W_a}=\begin{cases}a-x, & x<a\\ 0, & x\ge a\end{cases}$$

$$F^{(3)}_{W_a}=\begin{cases}0, & x<a\\ (x-a)^2/2, & x\ge a\end{cases}, \qquad S^{(3)}_{W_a}=\begin{cases}(a-x)^2/2, & x<a\\ 0, & x\ge a\end{cases}$$

Figure 2 presents graphs of integral distribution function for $W_0$ for $k=1,2$.

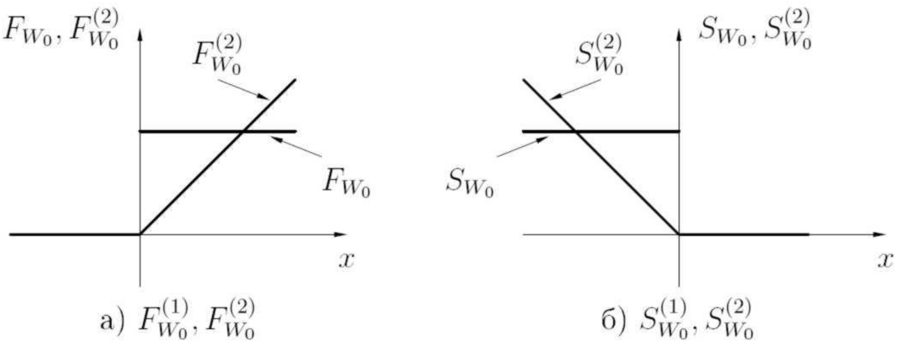



Figure 2. Graphs of integral distribution functions for $W_0$

Next consider Bernoulli distributions $B_{a,b,p}$ with parameters $-\infty<a<b<\infty$ and $0\le p\le 1$ and distribution functions

$$F_{B_{a,b,p}}(x)=\begin{cases}0, & x<a\\ 1-p, & a\le x<b\\ 1 & x\ge b\end{cases} \qquad S_{B_{a,b,p}}(x)=\begin{cases}1, & x<a\\ p, & a\le x<b\\ 0 & x\ge b\end{cases}$$

A few moments of this distribution are $\mu_0=1$, $\mu_1=a(1-p)+bp$, $\mu_2=a^2(1-p)+b^2p$, second order integral distribution functions have the form

$$F^{(2)}_{B_{a,b,p}}(x)=\begin{cases}0, & x<a\\ (1-p)(x-a), & a\le x<b\\ (1-p)(b-a)+(x-b), & x\ge b\end{cases}$$

$$S^{(2)}_{B_{a,b,p}}(x)=\begin{cases}0, & x\ge b\\ p(b-x), & a\le x<b\\ p(b-a)+(a-x), & x<a\end{cases}$$

and their graphs are depicted in figure 3.

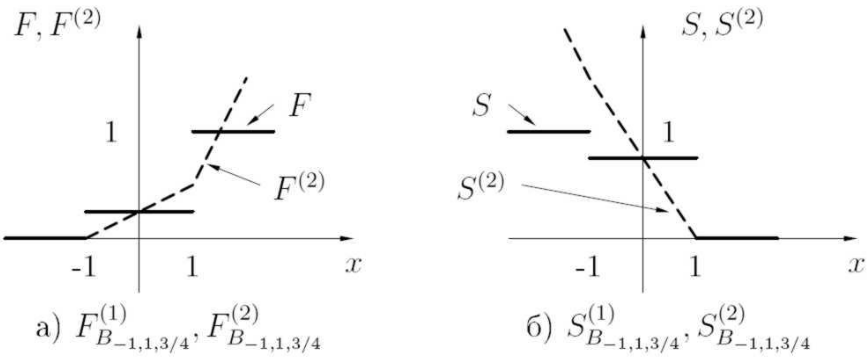



Figure 3. Graphs of integral distribution functions for $B_{-1,1,3/4}$

**Sense of stochastic dominance**

We say that there is an order $k$ *stochastic dominance of the first kind* (*direct stochastic dominance*) between distributions $P, Q \in \mathbf{P}$ (denoted $P \leq_k Q$) if

$$F_P^{(k)}(x) \geq F_Q^{(k)}(x), \qquad x \in \mathbf{R}. \tag{3}$$

Order $k$ *stochastic dominance of the second kind* (*dual stochastic dominance*, denoted $P \leq_k^* Q$) if

$$S_P^{(k)}(x) \leq S_Q^{(k)}(x), \qquad x \in \mathbf{R}. \tag{4}$$

Note that inequality signs in (3) and (4) are opposite. If the distributions $P, Q$ are generated by random variables $X, Y$ respectively, we will naturally define stochastic dominance between random variables, and use notation $X \leq_k Y$ and $X \leq_k^* Y$ for direct and dual dominance, respectively.

Let $X$ be a random variable with finite expectation, which generates a distribution $P$ on $(\mathbf{R}, \mathbf{B})$ with distribution function $F$ and complementary distribution function $S$. We will call $\pi(X) = \mathbf{E}X$ *pure premium* for risk $X$ transfer. One can easily see that

$$\mathbf{E}X = t + S^{(2)}(t) - F^{(2)}(t), \qquad t \in \mathbf{R}.$$

Denote $(x)_+ = \max(x, 0)$. For a given risk $X$ fix a number $K \in \mathbf{R}$, consider risk $(X - K)_+ = \max(X - K, 0)$, and calculate its pure premium $\mathbf{E}(X - K)_+$. We have

$$\mathbf{E}(X-K)_+ = \int_{-\infty}^{\infty} \max(x-K,0)\,dF(x) = \int_K^{\infty} (x-K)\,dF(x) =$$

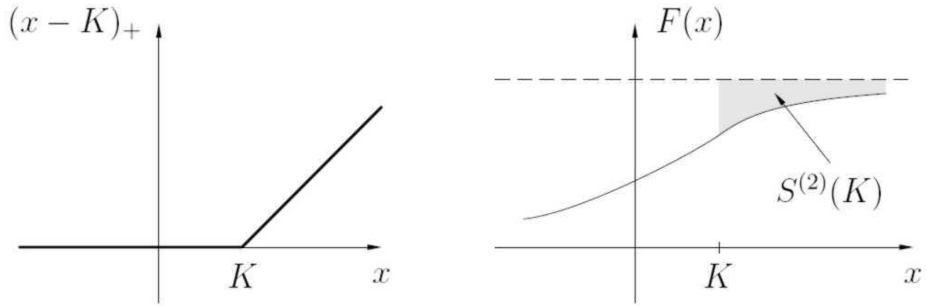
$$= \int_K^{\infty} (K-x)\,d(1-F(x)) = (K-x)(1-F(x))\,|_K^{\infty} + \int_K^{\infty} (1-F(x))\,dx = S^{(2)}(K).$$

For pure premium of risk $(K - X)_+$ similarly obtain $\mathbf{E}(K - X)_+ = F^{(2)}(K)$, so

$$\mathbf{E}(K-X)_+ = F^{(2)}(K), \qquad \mathbf{E}(K-X)_+ = F^{(2)}(K). \tag{5}$$

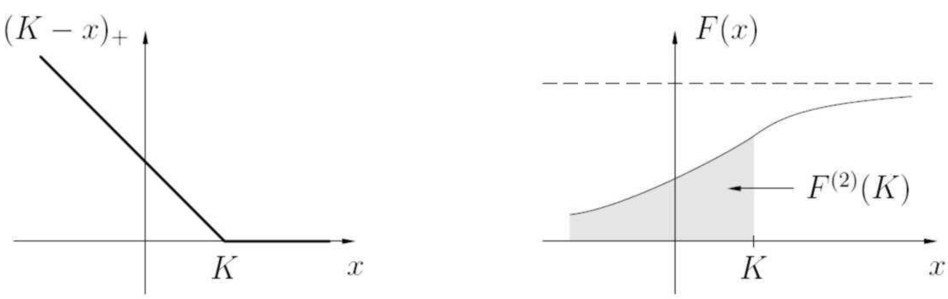
Consider an asset for which the price at some future moment $T$ is described by a random variable $X$. Denote $F$ and $S$ direct and complementary distribution functions of $X$, respectively. European call option on the asset with strike price $K$ and maturity $T$ possess the payoff function of the form $u_K(X) = (X-K)_+$. Pure premium for the option according to (5) has the form $\pi(u_K(X)) = S^{(2)}(K)$. Payoff function for a European put option on the same asset and similar parameters $K, T$ has the form $l_K(X) = (K-X)_+$. Pure premium for this option is $\mathbf{E}l_K(X) = F^{(2)}(K)$. Thus second order stochastic dominance $X \leq_2 Y$ implies that pure premium for put option on the asset $X$ is not greater than pure premium for put option on $Y$ for any strike price $K$. Dual stochastic dominance $X \leq_2^* Y$ implies similar relation for call option on $X$ and $Y$.

Graphs of payoff functions $u_K(x) = (x-K)_+$ and $l_K(x) = (K-x)_+$, and pure premiums $\mathbf{E}u_K(X) = S^{(2)}(K)$, $\mathbf{E}l_K(X) = F^{(2)}(K)$ are presented in figures 4 and 5.




a) payoff function $u_K$ b) $S^{(2)}(K)$

Figure 4. Payoff function and pure premium for a European call option




a) payoff function $l_K$ b) $F^{(2)}(K)$

Figure 5. Payoff function and pure premium for a European put option

For a given level $K$ consider a sharing of risk $X$ of the form

$$X = L_K(X) + U_K(X), \tag{6}$$

where the random variables $L_K(X), U_K(X)$ are derived from $X$ by the following non-decreasing functions

$$L_K(x) = \min(x, K) = \begin{cases} x, & x \le K \\ K, & x > K \end{cases}, \qquad U_K(x) = \max(0, x-K) = (x-K)_+ = \begin{cases} 0, & x \le K \\ x-K, & x > K \end{cases}.$$

Clearly $U_K(X) = (X-K)_+$, $L_K(X) = K - (K-X)_+$, so

$$\pi(U_K(X)) = \mathbf{E}U_K(X) = S^{(2)}(K), \qquad \pi(L_K(X)) = \mathbf{E}L_K(X) = K - F^{(2)}(K).$$

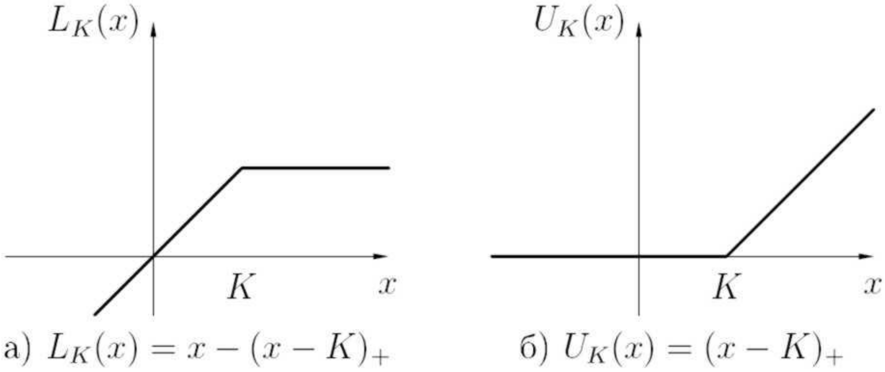



Figure 6. Risk layers in stop-loss reinsurance

The sharing just described appears for example in stop-loss reinsurance, where $L_K(X)$ stands for retention and $U_K(X)$ is transferred to reinsurer. Thus dual second order stochastic dominance $X \le_2^* Y$ implies $\pi(U_K(X)) \le \pi(U_K(Y))$ for any $K \in \mathbf{R}$, in other words, pure premium for reinsurance of the risk $X$ if not greater that of the risk $Y$ . Risk $Y$ appears to be more expensive for reinsurance.

A similar risk sharing appears in franchise insurance, where dual stochastic dominance implies that franchise insurance of $X$ is cheaper than that of $Y$ for any franchise level $K \in \mathbf{R}$.

**References**

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