

Measuring Risk Aversion

In Natural Preference Systems

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Risks Space

Basic Concepts

- $(\Omega, \mathcal{A}, \mathbf{P})$: probability space
- $X : \Omega \rightarrow \mathbf{R}$: random variable (risk)
- \mathcal{X} : set of risks endowed with a norm $\| \cdot \|$
- $I \in \mathcal{X} : I(\omega) = 1, \omega \in \Omega$
- Order: $X \leq Y$ means $X(\omega) \leq Y(\omega), \omega \in \Omega$
- Positive Cone $C_+ = \{X \in \mathcal{X} : X \geq 0\}$

Risks Space

Norm examples

- $\|X\|_p = (\int_{\Omega} |X(\omega)|^p d\mathbf{P}(\omega))^{1/p}$
- $\|X\|_{\infty} = \sup_{\omega \in \Omega} |X(\omega)|$
- Energy norms
- Minkowski norms

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Preference Relation

Axioms and Representation

- Axioms
 - Completeness: $X \succeq Y$ or $Y \succeq X$ **for each pair** $X, Y \in \mathcal{X}$
 - Transitivity: $X \succeq Y, Y \succeq Z \implies X \succeq Z$
- Parts
 - Symmetric part $X \sim Y \iff X \succeq Y, Y \succeq X$
 - Asymmetric part $X \prec Y \iff X \succeq Y, Y \not\succeq X$
- Representation
 - $X \succeq Y \iff f(X) \leq f(Y)$

Preference Relation

Classes

- Classes
 - Upper class $X \in \mathcal{X} \implies U(X) = \{Y : X \preceq Y\}$
 - Lower class $X \in \mathcal{X} \implies L(X) = \{Y : Y \preceq X\}$
 - Equivalence class

$$X \in \mathcal{X} \implies K(X) = U(X) \cap L(X) = \{Y : Y \sim X\}$$
- Mapping
 - Partitioning $C : \mathcal{X} \rightarrow \mathcal{X} / \sim; X \mapsto K(X)$

Preference Relation

Properties

- \preceq is **Finite** if $\forall Y \in \mathcal{X} \exists a < b : al \in L(Y), bl \in U(Y)$
- \preceq is **Monotone** if $X \leq Y \implies X \preceq Y$
- \preceq is **Strictly Monotone** if $X \leq Y, X \neq Y \implies X \prec Y$
- \preceq is **Lower Semicontinuous at** $Y \in \mathcal{X}$ if it is monotone and for any countable set $A \subseteq U(Y)$ we have $\inf_{X \in A} X \in U(Y)$
- \preceq is **Upper Semicontinuous at** $Y \in \mathcal{X}$ if it is monotone and for any countable set $A \subseteq L(Y)$ we have $\sup_{X \in A} X \in L(Y)$
- \preceq is **Continuous** if it is Lower and Upper Semicontinuous at each $Y \in \mathcal{X}$
- \preceq is **Unsaturated** if $\forall X, Y \in \mathcal{X} \exists c, d \in \mathbf{R} : X + cl \prec Y \prec X + dl$

Preference Relation

Natural Preference

Definition 1

Continuous, finite, strictly monotone, unsaturated preference relation is called **natural**

Theorem 1

Let \preceq be natural. Then for any $X \in \mathcal{X}$ we have

$$U(X) = \bigcup_{Y \in K(X)} (Y + C_+)$$

Theorem 2

Let \preceq be natural. Then there is a representative functional $f : \mathcal{X} \rightarrow \mathbf{R}$; in particular it may be chosen to satisfy

$$f(cl) = c, \quad c \in \mathbf{R}$$

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Expected utility

Risk aversion in the small

- Given an utility function $U : \mathbf{R} \rightarrow \mathbf{R}$
- Expected utility functional $u(X) = \mathbf{E}(U(X))$, $X \in \mathcal{X}$
- $A(x) = -\frac{U''(x)}{U'(x)}$, $x \in \mathbf{R}$

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Risk Aversion

Concept

- Distortions $\mathcal{X}_0 = \{X \in \mathcal{X} : \mathbf{E}X = 0\}$
- Qualitative Risk Aversion $xI + \Delta \prec xI, \Delta \in \mathcal{X}_0, x \in \mathbf{R}$
- Norm calibration $\|I\| = 1$
- Unit Ball $B = \{X \in \mathcal{X} : \|X\| \leq 1\}$
- Unit Ball Section $B_0 = \mathcal{X}_0 \cap B$
- Risk Aversion Function

$$r(x) = \sup_{\Delta \in B_0} (f(xI) - f(xI + \Delta)), x \in \mathbf{R}$$

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Theorem 3

Let \preceq be natural, represented by f . Then

$$r(x) = x - \sup\{y : U(y) \succeq xI + B_0\}, \quad x \in \mathbf{R}$$

Theorem 4

Let \preceq_1, \preceq_2 be natural, and $U_1(xI) \subseteq U_2(xI)$, $x \in \mathbf{R}$. Then $r_1(x) \geq r_2(x)$, $x \in \mathbf{R}$.

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Energy Norm

- Energy norm $\|X\|_A = \sqrt{X'AX}$
- Simple space $|\Omega| = 2$, $\mathbf{P}(\omega) = 0.5$, $\omega \in \Omega$

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad a = \frac{1+c}{4}, \quad b = \frac{1-c}{4}, \quad c > 0$$

- $r(x) = 1/\sqrt{c}$

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Coherent Risk Measure

$$f(X) = \inf_{Q \in \mathcal{Q}} \mathbf{E}_Q X$$

- Simple space $|\Omega| = 2$, $\mathbf{P} = \left(\frac{5}{8}, \frac{3}{8}\right)$,
- Norm $\|X\| = \max_i |X_i|$
- $\mathcal{Q} = \left\{ \left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{3}{4}, \frac{1}{4}\right) \right\}$

$$\mathcal{X}_0 : \frac{5}{8}X_1 + \frac{3}{8}X_2 = 0, \quad \partial B_0 = \left\{ \left(\frac{3}{5}, -1\right), \left(-\frac{3}{5}, 1\right) \right\}$$

$$r = \max_{Y \in \partial B_0} (|f(Y)|) = \max \left(\frac{1}{5}, \frac{7}{15} \right) = \frac{7}{15}$$

Acknowledgements

Thank you so much for not beating me to death!